ON THE HOMOGENEIZED WEYL ALGEBRA

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Abstract. The aim of this paper is to give relations between the category of finetely generated graded modules over the homogeneized Weyl algebra B_n, the finetely generated modules over the Weyl algebra A_n and the finetely generated graded modules over the Yoneda algebra $B_n^!$ of B_n. We will give these relations both at the level of the categories of modules and at the level of the derived categories.

1. Various algebras associated to A_n

We assume through the paper that the reader is familiar with basic results on Weyl algebras, as in the book by Coutinho [Co] and the notes by Miličić [Mil], as well as, with basic results on Koszul algebras [GM1], [GM2], and derived categories, for which we will refer to Miyachi's notes [Mi].

We begin the paper stating without proofs some basic results on the homogenized Weyl algebras and refer the reader to [Mo] for the proofs.

Let K be a field of zero characteristic, it is well known that the Weyl algebra A_n has the following description by generators and relations:

$A_n = K < X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n > /\{[X_i, \delta_j] = \partial_{ij}, [X_i, X_j`, [\delta_i, \delta_j]\}$, where $K < X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n >$ is the free algebra in $2n$ generators and $[X, Y`]$ denotes the commutator $XY - YX$.

Several filtrations can be given to A_n but it is not graded by path length. We will associate to A_n a quadratic algebra the so called homogenized Weyl algebra defined by quiver and relations as follows: $B_n = K < X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n, Z >$ $/\{[X_i, \delta_j] = \partial_{ij}Z^2, [X_i, X_j`, [\delta_i, \delta_j], [X_i, Z`], [\delta_i, Z`\}$.

The algebras B_n are related to the Weyl algebras as follows: Take the quotients $A_{n,c} = B_n/\{Z - c\}$ with $c \in K$.

When $n = 0$ the algebra $A_{n,0}$ is isomorphic to the polynomial algebra $K[X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n]$ and $A_{n,1}$ is isomorphic to A_n for $c \neq 0$, and K algebraically closed the algebras $A_{n,c}$ are all isomorphic to A_n.

By construction, the polynomial algebra $K[z]$ is contained in the center of B_n.

The family of monomials of B_n, $\{Z^iX^J\delta^L \mid i \geq 0,\ J, L \in N^n\}B_n$ is a Poincare-Birkoff basis.

An element $b \in B_n$ can be written as $b = \sum b_{i,P,Q}Z^iX^P\delta^Q$ define $\partial(b) = \max\{\mid P \mid + \mid Q \mid\mid b_{i,P,Q} \neq 0\}$

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1) $\partial(a+b) \leq \max\{\partial(a), \partial(b)\}$
2)$\partial(ab) = \partial(a) + \partial(b)$

Lemma 1. *The center of B_n is isomorphic to $K[Z]$.*

Proof. Let $b \in Z(B_n)$ be an element of the center,
$b = \sum\limits_{n=0}^{m}(\sum\limits_{k+|\alpha|+|\beta|=n} a_{k,\alpha,\beta} Z^k X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n} \delta_1^{\beta_1} \delta_2^{\beta_2} ... \delta_n^{\beta_n})$.
We have an equality $\delta_1 X_1 = X_1 \delta_1 + Z^2$, which implies
$\delta_1 X_1^2 = X_1 \delta_1 X_1 + Z^2 X_1 = X_1(X_1\delta_1 + Z^2) + Z^2 X_1 = X_1^2 \delta_1 + 2Z^2 X_1$.
It follows by induction $\delta_1 X_1^{\alpha_1} = X_1^{\alpha_1}\delta_1 + \alpha_1 Z^2 X_1^{\alpha_1 - 1}$.
Hence $\delta_1 Z^k X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n} \delta_1^{\beta_1} \delta_2^{\beta_2} ... \delta_n^{\beta_n} =$
$Z^k X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n} \delta_1^{\beta_1+1} \delta_2^{\beta_2} ... \delta_n^{\beta_n} + \alpha_1 Z^{k+2} X_1^{\alpha_1-1} X_2^{\alpha_2} ... X_n^{\alpha_n} \delta_1^{\beta_1} \delta_2^{\beta_2} ... \delta_n^{\beta_n}$ and
$Z^k X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n} \delta_1^{\beta_1} \delta_2^{\beta_2} ... \delta_n^{\beta_n} \delta_1 = Z^k X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n} \delta_1^{\beta_1+1} \delta_2^{\beta_2} ... \delta_n^{\beta_n}$
From the equality $\delta_1 b = b\delta_1$, after cancellation we have
$(\sum\limits_{k+|\alpha|+|\beta|=n} \alpha_1 a_{k,\alpha,\beta} Z^{k+2} X_1^{\alpha_{1-1}} X_2^{\alpha_2} ... X_n^{\alpha_n} \delta_1^{\beta_1} \delta_2^{\beta_2} ... \delta_n^{\beta_n}) = 0$.
It follows $\alpha_1 = 0$.
Multiplying by: $\delta_2, ...\delta_n$,we get $b = \sum\limits_{n=0}^{m}(\sum\limits_{k+|\beta|=n} a_{k,\beta} Z^k \delta_1^{\beta_1} \delta_2^{\beta_2} ... \delta_n^{\beta_n})$ and multiplying by: $X_1, X_2, ...X_n$ we obtain by a similar calculation $b = \sum\limits_{k=0}^{m} a_k Z^k$. □

1.1. **A Filtration on B_n.**. Define a filtration on B_n as follows: $F_t B_n = \{b \in B_n \mid \partial(b) \leq t\}$

It has the following properties:
0. $F_0 B_n = K[Z]$
1. $F_t B_n = 0$ for n<0
2. $\bigcup\limits_{n \in Z} F_t B_n = B_n$
3. 1$\in F_0 B_n$
4. $F_p B_n F_t B_n \subset F_{p+t} B_n$

5. For any pair of integers, p,t and elmernts $a \in F_p B_n$ and $b \in F_t B_n$ the commutator $[a,b] = ab - ba$ is in $F_{p+t-1} B_n$.
6. $GrB_n \cong K[Z, X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n]$
7. $Gr_1 B_n$ is generated as K-vector space by $Z, X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n$.

These conditions imply $\{F_t B_n\}$ is a "good filtration" (Miličić) as a consequence we get:

Proposition 1. *B_n is noetherian as left and right ring.*

Proposition 2. *The ring B_n has global dimension $2n+1$.*

Proposition 3. *B_n has Gelfand-Kirillov dimension $2n+1$.*

Theorem 1. *Let M be a finitely generated B_n-module, denote by $d_\lambda(M)$ the Gelfand-Kirillov dimension of M. Then:*
i) $Ext^i_{B_n}(M, B_n) = 0$ $fori < 2n+1$-$d_\lambda(M)$
ii) $d_\lambda(Ext^i_{B_n}(M, B_n)) \leq 2n+1-i$ for all $0 \leq i \leq 2n+1$
iii) $d_\lambda(Ext^{2n+1-d_\lambda(M)}_{B_n}(M, B_n)) = d_\lambda(M)$

Corollary 1. *The algebra B_n is Artin Schelter regular.*

Since B_n has a Poincare-Birkoff basis and it is quadratic by [Li], [GH] it is Koszul. Let $B_n^!$ be its Yoneda algebra $B_n^! = \underset{k\geq 0}{\oplus} Ext_B^k(K,K)$, by [Sm] $B_n^!$ is selfinjective. It follows by general properties of Koszul algebras [GM1] ,[GM2] that $B_n^!$ has the same quiver as B_n and relations orthogonal with respect to the canonical bilinear form. It is easy to see that $B_n^!$ has the following form:

$B_n^! = K_q[X_1,\, X_2,\, ...X_n,\, \delta_1,\, \delta_2, ...\delta_n,\, Z]/\{X_i^2,\, \delta_j^2,\, \sum\limits_{i=0}^{n} X_i\delta_i + Z^2\}$,

where $K_q[X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n, Z]$ denotes the quantum polynomial ring.

$K < X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n, Z > /\{(X_i, X_j), (\delta_i, \delta_j), (X_i, Z), (\delta_i, Z)\}$. Here (X, Y) denotes the anti commutators $XY + YX$.

The polynomial algebra $C_n = K[X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n, Z]$ is a Koszul algebra with Yoneda algebra the exterior algebra:

$C_n^! = K_q[X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n, Z]/\{X_i^2, \delta_j^2\}$.

Observe we obtain C_n as a quotient of B_n and $C_n^!$ is a sub algebra of $B_n^!$. We want to prove $B_n^!$ is a free $C_n^!$-module of rank two.

Lemma 2. *The quantum polynomial algebra $K_q[X_1, X_2, ...X_m]$ has a Poincare-Birkoff basis, in particular it is a noetherian Koszul algebra.*

Proof. It is easy to see $K_q[X_1, X_2, ...X_m]$ has a quadratic Groebner basis then by [Li] it has a Poincare-Birkoff basis hence it is noetherian and by [GH] it is Koszul. □

It is clear that $B_n^!$ is generated as K- vector space by square free words: $X_{j_1} X_{j_2} ...X_{j_t}\delta_{i_1}\delta_{i_2}...\delta_{i_s}$, $X_{j_1}\ X_{j_2}\ ...X_{j_t}\delta_{i_1}\delta_{i_2}...\delta_{i_s}Z$ with $X_{j_u} \neq X_{j_v}$ for $u \neq v$ and $\delta_{i_{k1}} \neq \delta_{i\ell}$ for $k \neq \ell$. We then have $B_n^! = C_n^! + ZC_n^!$.

Proposition 4. *There exists a $C_n^!$-module decomposition: $B_n^! = C_n^! \oplus ZC_n^!$.*

Proof. Since $B_n^!$ is graded by path length, we can consider only linear combinations of paths (words) of the same length. We will use the standard notation: $X^\alpha = X^{\alpha_1}.X^{\alpha_2}...X^{\alpha_n}$ and $\delta^\beta = \delta^{\beta_1}\delta^{\beta_2}...\delta^{\beta_n}$ where $\alpha_i, \beta_j \in \{1, 0\}$ and $\mid \alpha \mid= \sum\limits_{i=1}^{n}\alpha_i$, $\mid \beta \mid= \sum\limits_{i=1}^{n}\beta_i$.

To see that the sum is direct we consider linear combinations: $\sum\limits_{|\alpha|+|\beta|=m} \overline{c_{\alpha,\beta}X^\alpha\delta^\beta} + \sum\limits_{|\alpha'|+|\beta'|=m-1} \overline{c_{\alpha,\beta}X^{\alpha'}\delta^{\beta'}Z} = \overline{0}$.

This is: $\sum\limits_{|\alpha|+|\beta|=m} c_{\alpha,\beta}X^\alpha\delta^\beta + \sum\limits_{|\alpha'|+|\beta'|=m-1} c_{\alpha',\beta'}X^{\alpha'}\delta^{\beta'}Z = \sum\limits_{i=1}^{n} q_iX_i^2 + \sum\limits_{i=1}^{n} q_i'\delta_i^2 + p'(\sum\limits_{i=1}^{n} X_i\delta_i + Z^2) + p''(\sum\limits_{i=1}^{n} X_i\delta_i + Z^2)$

Where $p' = \sum\limits_{i=1}^{t} m_i'$, $p'' = \sum\limits_{i=1}^{s} m_i''$and m_i' are monomials divisible by a square element and $m_i''$are square free.

Let´s assume $c_{\alpha,\beta} \neq 0$.Comparing both terms of the equality, the term $X^\alpha\delta^\beta$is square free then $c_{\alpha,\beta}X^\alpha\delta^\beta$ does not cancel with elements of the form: $q_iX_i^2$, $q_i'\delta_i^2$, $p'Z^2$, $p''Z^2$ or $p'X_i\delta_i$.Therefore $c_{\alpha,\beta}X^\alpha\delta^\beta = m_i''X_j\delta_j$ and $m_i'' = c_{\alpha,\beta}X^{\alpha-\alpha_j}\delta^{\beta-\beta_j}$.

Then $m_i''Z^2$does not cancel with elements of the form $X^\alpha\delta^\beta$, $X^{\alpha'}\delta^{\beta'}Z$, X_i^2, δ_i^2 or $m_j''X_i\delta_i$.

Therefore: $m''_i Z^2 = m'_j X_k \delta_k = c_{\alpha,\beta} X^{\alpha-\alpha_j} \delta^{\beta-\beta_j} Z^2$ and
$m'_j = c_{\alpha,\beta} X^{\alpha-\alpha_j-\alpha_k} \delta^{\beta-\beta_j-\beta_k} Z^2$ with $j \neq k$, because $X^\alpha \delta^\beta$ is square free.

Now $m'_j Z^2 = c_{\alpha,\beta} X^{\alpha-\alpha_j-\alpha_k} \delta^{\beta-\beta_j-\beta_k} Z^4$ can be canceled only with an element of the form $m'_s X_t \delta_t$ and $m'_s = c_{\alpha,\beta} X^{\alpha-\alpha_j-\alpha_k-\alpha_t} \delta^{\beta-\beta_j-\beta_k-\beta_t} Z^4$.

We continue by induction until getting an element of the form $c_{\alpha,\beta} Z^m$ that can not be canceled, contradicting the assumption $c_{\alpha,\beta} \neq 0$.

In a similar way we get a contradiction when assuming $c_{\alpha',\beta'} \neq 0$. □

We have proved $B_n^!$ is a free $C_n^!$-module of rank two, this fact will have interesting applications.

For completeness we include the proof of the following well known result.

Lemma 3. *The Weyl algebra A_n is isomorphic to $B_n/(z-1)B_n$.*

Proof. We have a morphism $\varphi : K < X_1, X_2, ...X_n, \delta_1, \delta_2, ...\delta_n, Z > \longrightarrow A_n$ given by: $\varphi(X_i) = X_i$, $\varphi(\delta_i) = \delta_i$, $\varphi(Z) = 1$.

It follows $\varphi([X_i, X_j]) = \varphi([\delta_i, \delta_j]) = 0$, for $i \neq j$ $\varphi([X_i, \delta_j]) = 0$ and $\varphi(X_i\delta_i - \delta_i X_i - Z^2) = 0$, $\varphi(X_i Z - Z X_i) = \varphi(\delta_i Z - Z\delta_i) = 0$.

The map φ induces a surjection $\overline{\varphi} : B_n \to A_n$ such that $(Z-1)B_n \sqsubseteq Ker\overline{\varphi}$.

Let $\gamma = \sum\limits_{i,\alpha,\beta} c_{i,\alpha,\beta} Z^i X^\alpha \delta^\beta$ be an element in $Ker\overline{\varphi}$. We write $Z^i = (Z-1)f_i(Z) + 1$.

Then $\gamma = \sum\limits_{i,\alpha,\beta} c_{i,\alpha,\beta}(Z-1)f_i(Z)X^\alpha\delta^\beta + \sum\limits_{i,\alpha,\beta} c_{i,\alpha,\beta} X^\alpha \delta^\beta$.

Hence, $\overline{\varphi}(\gamma) = \sum\limits_{\alpha,\beta} (\sum\limits_i c_{i,\alpha,\beta}) X^\alpha \delta^\beta = 0$, implies $\sum\limits_i c_{i,\alpha,\beta} = 0$.

Therefore $\gamma = \sum\limits_{i,\alpha,\beta} c_{i,\alpha,\beta}(Z-1)f_i(Z)X^\alpha\delta^\beta \in (Z-1)B_n$. □

The embedding $K[Z] \to B_n$ is a morphism of graded algebras.

1.2. **The Nakayama automorphism.** In this subsection we will recall some basic facts about selfinjective finite dimensional algebras, that will be needed in the particular situation we are considering. We refer to the paper by Yamagata [Y] for more details.

Let A be a finite dimensional selfinjective K-algebra, over a field. Denote by $D(A_A) = Hom_K(A, K)$ the standard bimodule. There is an isomorphism of left A-modules $\varphi : A \to D(A)$,which induces by adjunction a map $\beta' : A \otimes_A A \to K$. By definition, $\beta'(a \otimes b) = \varphi(b)(a)$.The composition: $A \times A \xrightarrow{p} A \otimes_A A \xrightarrow{\beta'} K$, $\beta = \beta' p$ is a non degenerated $A-$bilinear form, and $A \otimes_A A$ is the cokernel of the map $A \otimes_K A \otimes_K A \to A \otimes_K A$ given by $a \otimes b \otimes c \to ab \otimes c - a \otimes bc$. Let $\pi : A \otimes_K A \to A \otimes_A A$ be the cokernel map.

The map β' induces a map $\overline{\beta} : A \otimes_K A \to K$ by $\overline{\beta}(x \otimes y) = \beta'(y \otimes x)$. Hence $\overline{\beta}$ is also non degenerated. In consequence, there is a K-linear isomorphism $\psi : A \to D(A)$, given by $\psi(a_1)(a_2) = \overline{\beta}(a_2 \otimes a_1) = \beta(a_1 \otimes a_2) = \varphi(a_2)(a_1)$, set $\sigma = \psi^{-1}\varphi$.

There is a chain of equalities: $\beta(\sigma(y), x) = \beta(\psi^{-1}\varphi(y), x) = \psi\psi^{-1}\varphi(y)(x) = \varphi(y)(x) = \overline{\beta}(y \otimes x)$.

The map $\sigma : A \to A$ is an isomorphism of K-algebras.

$\beta(\sigma(y_1y_2) \otimes z) = \beta(z \otimes y_1y_2) = \beta(zy_1 \otimes y_2) = \overline{\beta}(y_2, zy_1) = \beta(\sigma(y_2), zy_1) = \beta(\sigma(y_2)z, y_1) = \overline{\beta}(y_1, \sigma(y_2)z) = \beta(\sigma(y_1), \sigma(y_2)z) = \beta(\sigma(y_1)\sigma(y_2), z)$.

Since z is arbitrary and β non degenerated $\sigma(y_1y_2) = \sigma(y_1)\sigma(y_2)$.

Let $D(A)_\sigma$ $({}_{\sigma^{-1}}D(A))$ be the A- A bimodule with right (left) multiplication shifted by σ (σ^{-1}). Then $\varphi : A \to D(A)_\sigma$, and $\psi : A \to_{\sigma^{-1}} D(A)$ are isomorphisms of A- A bimodules.

$\varphi(xa)(y) = \beta(y, xa) = \beta(\sigma(x)\sigma(a), y) = \beta(\sigma(x), \sigma(a)y) = \beta(\sigma(a)y, x) =$

$\varphi(x)(\sigma(a)y) = \varphi(x)\sigma(a)(y)$, for all y. Therefore $\varphi(xa) = \varphi(x)\sigma(a)$. As claimed.

In a similar way, $\psi(xb)(y) = \varphi(y)(xb) = \beta(xb, y) = \beta(x, by) = \varphi(by)(x) = \psi(x)b(y)$. Since y is arbitrary, $\psi(xb) = \psi(x)b$.

In the other hand, $\varphi(y)(x) = (\psi(x)(y) = \beta(x, y) = \beta(\sigma\sigma^{-1}(x), y) = \beta(y, \sigma^{-1}(x)) = \varphi(\sigma^{-1}(x))(y)$. Hence, $\psi(x) = \varphi(\sigma^{-1}(x))$.

It follows: $\psi(bx) = \varphi(\sigma^{-1}(b)\sigma^{-1}(x)) = \sigma^{-1}(b)\varphi(\sigma^{-1}(x)) = \sigma^{-1}(b)\psi(x)$.

Let M be a finitely generated A-module. Since ${}_\sigma A \cong D(A)$ as bimodule, there are natural isomorphisms:

$D(M^*) = Hom_A(Hom_A(M, A), D(A) = Hom_A(Hom_A(M, A),_\sigma A) \cong \sigma M^{**} \cong \sigma M$, where $\sigma M = M$ as abelian group and multiplication by A shifted by σ.

We look now to the case A a positively graded selfinjective K-algebra and $\varphi : A \to D(A)[n]$ an isomorphism of graded A-modules. Let a, x be elements of A of degrees k and j, respectively. Then $\varphi(xa) = \varphi(x)\sigma(a)$ is an homogeneous element of degree $k + j$ and $\sigma(a) = \Sigma\sigma(a)_i$, with $\sigma(a)_i$ homogeneous elements of degree i. Hence $\varphi(x)\sigma(a)_i = 0$ for all $i \neq j$. $\varphi(x)\sigma(a)_i(1) = \varphi(x)(\sigma(a)_i) = \beta(\sigma(a)_i, x) = 0$ for all homogeneous elements x. Hence $\beta(\sigma(a)_i, A) = 0$ and $\sigma(a)_i = 0$ for $i \neq j$.

We have proved σ is an isomorphism of graded K-algebras, which induces isomorphisms of graded A- A bimodules: $\varphi : A \to D(A)_\sigma[n]$, and $\psi : A \to_{\sigma^{-1}} D(A)[n]$.

We only need to check ψ is an isomorphism of graded K-vector spaces.

Being φ a graded map, $\varphi = \{\varphi_\ell\}$ and $\varphi_\ell : A_\ell \to Hom_K(A_{n-\ell}, K)[n]$ isomorphisms, inducing maps $\beta'_\ell : A_{n-\ell} \otimes_A A_\ell \to K$ and $\overline{\beta}_\ell : A_\ell \otimes_K A_{n-\ell} \to K$, with $\overline{\beta}_\ell(x \otimes y) = \beta'_\ell(y \otimes x)$.Each $\overline{\beta}_\ell$induces maps $\psi_\ell : A_{n-\ell} \to D(A_\ell)[n]$ such that $\psi = \{ \psi_\ell\}$ becomes a graded map.

In case M is a finitely generated graded left A-module there is a chain of isomorphisms of graded A-modules:

$D(M^*) = Hom_A(Hom_A(M, A), D(A)) = Hom_A(Hom_A(M, A),_\sigma A[-n]) \cong \sigma M^{**}[-n] \cong \sigma M[-n]$.

Assume now A is Koszul sefinjective with Nakayama automorphism σ and Yoneda algebra B. It was remarked in [M] that under these conditions there is natural action of σ as a graded automorphism of B, we will recall now this construction.

Let x be an element of $Ext^n_A(A_0, A_0) = \underset{i,j}{\oplus} Ext^n_A(S_i, S_j)$, x=$(x_{i,j})$ with $x_{i,j}$ the extension:

$$x_{i,j} : 0 \to S_j[-n] \to E_1 \to E_2 \to ... \to E_n \to S_i \to 0.$$

Then $\sigma x_{i,j}$ is the extension:

$$\sigma x_{i,j} : 0 \to \sigma S_j[-n] \to \sigma E_1 \to \sigma E_2 \to ... \to \sigma E_n \to \sigma S_i \to 0.$$

Since σ is a permutation of the graded simple, $\sigma x = (\ \sigma x_{i,j})$ is an element of $Ext^n_A(A_0, A_0)$ and $\sigma : Ext^n_A(A_0, A_0) \to Ext^n_A(A_0, A_0)$ is an isomorphism of K-vector spaces which extends to a graded automorphism of $B = \underset{n\geq 0}{\oplus} Ext^n_A(A_0, A_0)$.

Let M be a finitely generated (graded) A-module and $x = (x_j) \in Ext^n_A(M, A_0) = \underset{j\geq 0}{\oplus} Ext^n_A(M, S_j)$.

$$x_{,j} : 0 \to S_j[-n] \to E_1 \to E_2 \to ... \to E_n \to M \to 0.$$

Then $\sigma x = (\sigma x_j)$ with

$$\sigma x_{,j} : 0 \to \sigma S_j[-n] \to \sigma E_1 \to \sigma E_2 \to ... \to \sigma E_n \to \sigma M \to 0.$$

In this case there is an isomorphism of K-vector spaces: $\sigma : Ext^n_A(M, A_0) \to Ext^n_A(\sigma M, A_0)$, which induces a graded isomorphism: $\sigma : \underset{n\geq 0}{\oplus} Ext^n_A(M, A_0) \to \underset{n\geq 0}{\oplus} Ext^n_A(\sigma M, A_0)$.

We will also call the Nakayama automorphism to the automorphism σ of B.

We will look now in more detail to the Nakayama automorphism σ of $B^!_n$ of the shriek algebra of the homogenized algebra B_n.

The graded ring $B^!_n$ has a sum decomposition: $(B^!_n)_0 = (C^!_n)_0$, $(B^!_n)_1 = (C^!_n)_1 \oplus Z(C^!_n)_0$, $(B^!_n)_2 = (C^!_n)_2 \oplus Z(C^!_n)_1$, ... $(B^!_n)_i = (C^!_n)_i \oplus Z(C^!_n)_{i-1}$,... $(B^!_n)_{2n} = (C^!_n)_{2n} \oplus Z(C^!_n)_{2n-1}$, $(B^!_n)_{2n+1} = Z(C^!_n)_{2n}$

The algebra $C^!_n$ is the exterior algebra in $2n$ variables, hence,

$\dim_K(C^!_n)_j = \begin{pmatrix} 2n \\ j \end{pmatrix} = \begin{pmatrix} 2n \\ 2n-j \end{pmatrix} = \dim_K(C^!_n)_{2n-j}$.

Since $(B^!_n)_j = (C^!_n)_j \oplus Z(C^!_n)_{j-1}$, it follows:

$\dim_K(B^!_n)_j = \begin{pmatrix} \text{2n} \\ \text{j} \end{pmatrix} + \begin{pmatrix} \text{2n} \\ \text{j-1} \end{pmatrix} = \begin{pmatrix} \text{2n} \\ \text{2n+1-j} \end{pmatrix} + \begin{pmatrix} \text{2n} \\ \text{2n-j} \end{pmatrix} = \dim_K(B^!_n)_{2n+1-j}$.

The graded left module $D(\ (B^!_n)$ decomposes in homogeneous components:

$D(\ (B^!_n) = D(\ (B^!_n)_{2n+1}) + D(\ (B^!_n)_{2n}) + ...D(\ (B^!_n)_0)$.

Each component $(B^!_n)_i$ has as basis paths of length i either of the form:

$X_{j_1}X_{j_2}...X_{j_s}\delta_{j_s+1}\delta_{j_s+2}...\delta_{j_{i-1}}Z$ or $X_{j_1}X_{j_2}...X_{j_s}\delta_{j_s+1}\delta_{j_s+2}...\delta_{j_i}$ and $\mathrm{D((B^!_n)_{2n+1-i})}$ has as basis the dual basis of the ´paths of length $2n + 1 - i$.

The isomorphism of graded left modules: $\varphi : B^!_n \to D(\ B^!_n)[-2n - 1]$, sends a path of the form $\gamma = X_{j_1}X_{j_2}...X_{j_s}\delta_{j_s+1}\delta_{j_s+2}...\delta_{j_{i-1}}Z$ or of the form

$\gamma = X_{j_1}X_{j_2}...X_{j_s}\delta_{j_s+1}\delta_{j_s+2}...\delta_{j_i}$ to the dual basis $f_{\delta-\gamma}$ of the path $\delta-\gamma$ of length $2n + 1 - i$, with δ the path of maximal length $\delta = X_1X_2...X_n\delta_1\delta_2...\delta_nZ$.

Since $(B^!_n)_i = (C^!_n)_i \oplus Z(C^!_n)_{i-1}$ the isomorphisn ϕ restricts to isomorphisms of K-vector spaces $\varphi : (C^!_n)_i \to D(Z(C^!_n)_{2n-i-1})$ and $\varphi : (\ ZC^!_n)_{i-1} \to D((C^!_n)_{2n-i})$, hence, φ induces isomorphisms of $C^!_n$-modules $\varphi : (C^!_n) \to D(Z(C^!_n))$ and $\varphi : (\ ZC^!_n) \to D((C^!_n)$.

Now the isomorphism $\psi : B^!_n \to D(\ B^!_n)[-2n - 1]$ given $\psi(b_1)(b_2) = \varphi(b_2)(b_1)$ is such that for $c_1 \in (C^!_n)_i$ and $b \in B_n$, $b = \sum_{i=0}^{2n+1} b_i$, $\psi(c_1)(b) = \sum_{k=0}^{2n+1} \varphi(b_k)(c_1)$, since for all $k \neq i$ the length of b_k is different from $2n + 1 - i$, then $\varphi(b_k)(c_1) = 0$ and $\psi(c_1) \in D(Z(C^!_n)_{2n-i-1})$, hence, ψ induces an isomorphism of graded $C^!_n$-modules $\psi : (C^!_n) \to D(Z(C^!_n))$ and in a similar way an isomorphism $\psi : (\ ZC^!_n) \to D((C^!_n)$.It follows the Nakayama automorphism σ restricts to an automorphisms of graded rings: $\sigma : C^!_n \to C^!_n$ and of $C^!_n$-bimodules $\sigma : ZC^!_n \to ZC^!_n$.

Any automorphism σ of a ring B takes the center to the center, since $z \in Z(B)$ implies that for any $b \in B$, $\sigma(zb) = \sigma(z)\sigma(b) = \sigma(bz) = \sigma(b)\sigma(z)$.

In case B is the homogenized Weyl algebra $\sigma(Z)$ is an homogeneous element of degree one in $Z(B) = K[Z]$. Therefore $\sigma(Z) = kZ$ with k a non zero element of the field K.

Definition 1. *Let B be the homogenized Weyl algebra. A left B-module M is of Z-torsion if for any element m in M,there is a non negative integer k such that $Z^km = 0$.*

Let M be a Koszul left $B^!$-module such that $F(M) = \underset{n\geq 0}{\oplus} Ext^n_A(M, A_0)$ is a B-module of Z-torsion. Then $F(\sigma M)$ is of Z-torsion, in particular $F(D(M^*)[n])$ is of Z-torsion.

Since $F(\sigma M) = \sigma FM$ for $x \in F(M)$ there is an integer $k \geq 0$ such that $Z^k x = 0$ and in σFM, $Z^k * x = \sigma(Z^k)x = c^k Z^k x = 0$.

2. The graded localization of the homogenized Weyl algebra

Consider the multiplicative subset $S = \{1, Z, Z^2, ...\}$of $K[Z]$. The localization $K[Z]_S = \{f/z^n \mid n \geq 0\}$is a Z-graded algebra with homogeneous elements $Z^n/Z^m$of degree $n-m$. It is clear $K[Z]_S = K[Z, Z^{-1}]$, the Laurent polynomials.

The natural map: $K[Z] \to K[Z, Z^{-1}]$ is flat.

Our aim in this section is to study the graded localization:

$(B_n)_Z = B_n \underset{K[Z]}{\otimes} K[Z, Z^{-1}]$. It is a Z-graded K-algebra with homogeneous elements $b/Z^k$of degree(b/Z^k)= degree(b)$-k$.

We will study this algebra and its relations with the Weyl algebra A_n.

The natural map $\varphi : B_n \to (B_n)_Z$ given by $b \to b \otimes 1$is a morphism of graded algebras.

Since Z is in the center of B_n the ideal $(Z-1)B_n$ is two sided. Let's consider the composition:

$B_n \xrightarrow{\varphi} (B_n)_Z \xrightarrow{\pi} (B_n)_Z/(Z-1)(B_n)_Z$.

Let b be an element of $Ker\pi\varphi$. Then $b/1 \in (Z-1)(B_n)_Z$ implies $b/1 = (Z-1)b'/Z^k$, hence there exist $t, \ell \geq 0$ such that $Z^\ell b = Z^t(Z-1)b' = (Z-1)g(Z)b + b$ and $b = (Z-1)(b\'Z^t - g(Z)b) \in (Z-1)B_n$.It follows $Ker\pi\varphi = (Z-1)B_n$ and we have a commutative diagram:

$$\begin{array}{ccccc} B_n & \xrightarrow{\varphi} & (B_n)_Z & \xrightarrow{\pi} & (B_n)_Z/(Z-1)(B_n)_Z \\ & q \searrow & & \nearrow \psi & \\ & & B_n/(Z-1)B_n & & \end{array}$$

where ψ is an injective ring morphism. We will prove it is an isomorphism.

Let $b/Z^k + (Z-1)(B_n)_Z$be an element of $(B_n)_Z/(Z-1)(B_n)_Z$. We re write b/Z^k as follows:

$b/Z^k = b_\ell/Z^\ell + b_{\ell-1}/Z^{\ell-1} + ...b_0 + b_1 Z + ...b_m Z^m$.

But we can re write $b_i/1$ as $b_i/1 = Z^i b_i/Z^i = (Z-1)g(Z)b_i/Z^i + b_i/Z^i$. Similarly, $b_j Z^j = b_j(Z-1)g(Z) + b_j$.

Then $b_i/1 + (Z-1)(B_n)_Z = b_i/Z^i + (Z-1)(B_n)_Z$ and $b_i Z^i/1 + (Z-1)(B_n)_Z = b_i/1 + (Z-1)(B_n)_Z$.

Therefore: $b/Z^k + (Z-1)(B_n)_Z = (b_\ell + b_{\ell-1} + ...b_0 + b_1 + ...b_m)/1 + (Z-1)(B_n)_Z$.

We have proved ψ is an isomorphism.

In fact we proved the following:

Lemma 4. *With the above notation, there exists ring isomorphisms:*
$B_n/(Z-1)B_n \cong (B_n)_Z/(Z-1)(B_n)_Z \cong A_n$.

Denote by τ the composition $\psi^{-1}\pi$, then the following triangle commutes:

$$\begin{array}{ccc} B_n & \xrightarrow{\varphi} & (B_n)_Z \\ & q \searrow \quad \swarrow \tau & \\ & B_n/(Z-1)B_n & \end{array}.$$

Since $(B_n)_Z$ is Z-graded we have the inclusion $((B_n)_Z)_0 \to (B_n)_Z$ and the projection $(B_n)_Z \longrightarrow (B_n)_Z/(Z-1)(B_n)_Z$.

Let $\theta : ((B_n)_Z)_0 \longrightarrow (B_n)_Z/(Z-1)(B_n)_Z$ be the composition.

Proposition 5. *The map $\theta : ((B_n)_Z)_0 \longrightarrow (B_n)_Z/(Z-1)(B_n)_Z$ is an isomorphism.*

Proof. We prove first θ is injective. Let $\hat{b}$ be an element of $((B_n)_Z)_0$. It can be written as $\hat{b} = \sum_{i=0}^{m} g_i(X,\delta)Z^{-n_i}$ with $g_i(X,\delta)$homogeneous polynomials of degree n_i and $n_0 > n_1 > ...n_m$.

We have the following equalities: $\hat{b}$=$Z^{-n_0}\sum_{i=0}^{m} g_i(X,\delta)Z^{n_0-n_i}$ and $\sum_{i=0}^{m} g_i(X,\delta)Z^{\mathrm{n_0\text{-}n_i}}$ $=g_0(X,\delta)+g_1(X,\delta)...g_m(X,\delta)+(Z-1)h(Z)g'(X,\delta)$.

Therefore: $\hat{b} = Z^{-n_0}\sum_{i=0}^{m} g_i(X,\delta) + (Z-1)Z^{-n_0}b'$.

Hence, $\theta(\hat{b}) = 0$ means $Z^{-n_0}\sum_{i=0}^{m} g_i(X,\delta) \in (Z-1)(B_n)_Z$.

There exists $s, t \geq 0$ such that $Z^t(\sum_{i=0}^{m} g_i(X,\delta)) = Z^s(Z-1)b''$.

Set $g(X,\delta) = \sum_{i=0}^{m} g_i(X,\delta)$, then $Z^t g(X,\delta) = g(X,\delta) + (Z-1)h(Z)g(X,\delta)$ and $g(X,\delta) = (Z-1)\overline{b}$ with $\overline{b} \in B_n$.

Hence $\overline{b} = b_0(X,\delta) + b_1(X,\delta)Z + ...b_k(X,\delta)Z^k$.

Then $g(X,\delta) = -b_0(X,\delta) + (b_0(X,\delta) - b_1(X,\delta))Z + ...(b_{k-1}(X,\delta) - b_k(X,\delta))Z^k + b_k(X,\delta)Z^{k+1}$

It follows $g(X,\delta) = -b_0(X,\delta)$ and $b_0(X,\delta) = b_1(X,\delta) = b_2(X,\delta) = ...b_k(X,\delta) = 0$

Therefore: $g(X,\delta) = \sum_{i=0}^{m} g_i(X,\delta) = 0$ but each $g_i(X,\delta)$ has degree n_i with $n_0 > n_1 > ...n_m$.

It follows each $g_i(X,\delta) = 0$ and $\hat{b} = 0$.

We prove now θ is surjective.

Take an element $b/Z^k + (Z-1)(B_n)_Z$ in $(B_n)_Z/(Z-1)(B_n)_Z$.

The element b decompose into homogeneous components: $b = \sum_{i=0}^{m} b_i$ with degree $b_i = n_i$ and $n_0 < n_1 < ...n_m$.

As above, $Z^{n_m-n_i}b_i = (Z-1)h(Z)b_i + b_i$. Set $b'_i = Z^{n_m-n_i}b_i$, $b\prime = \sum_{i=0}^{m} b'_i$.

Hence $b_i/Z^k + (Z-1)(B_n)_Z = b'_i/Z^k + (Z-1)(B_n)_Z$.

If $\ell > k$, then $b'/Z^k + (Z-1)(B_n)_Z = Z^{\ell-k}b'/Z^\ell = b'/Z^\ell + (Z-1)f(Z)b''$ and $b/Z^k + (Z-1)(B_n)_Z = b'/Z^\ell + (Z-1)(B_n)_Z$.

The case $\ell < k$ is similar: $b'/Z^\ell = Z^{k-\ell}b'/Z^k = b'/Z^k + (Z-1)f(Z)b''$and $b/Z^k + (Z-1)(B_n)_Z = b'/Z^\ell + (Z-1)(B_n)_Z$.

In any case $\theta(b'/Z^\ell) = b/Z^k + (Z-1)(B_n)_Z$.

We have proved θ is an isomorphism. □

With the identification $B_n/(Z-1)B_n \cong (B_n)_Z/(Z-1)(B_n)_Z \cong A_n$ we have proved the graded algebra $(B_n)_Z$ has A_n in degree zero.

Theorem 2. *There exists a graded rings isomorphism:* $A_n \underset{K}{\otimes} K[Z, Z^{-1}] \longrightarrow B_n \underset{K[Z]}{\otimes} K[Z, Z^{-1}]$.

Proof. Since $((B_n)_Z)_0 \cong A_n$, we will prove that multiplication induces a ring isomorphism: $\mu : ((B_n)_Z)_0 \underset{K}{\otimes} K[Z, Z^{-1}] \longrightarrow (B_n)_Z : b/Z^\ell \otimes Z^k \longrightarrow bZ^k/Z^\ell$, assume $b/Z^\ell$is homogeneous with degree $b = \ell + k$, then $b/Z^\ell = b/Z^{\ell+k}.Z^k$ and degree$(b/Z^{\ell+k}) = 0$.

Then $b/Z^{\ell+k} \otimes Z^k \longrightarrow b/Z^\ell$, similarly if degree $b = \ell - k$, $b/Z^{\ell-k} \otimes Z^k \longrightarrow b/Z^\ell$.The map μ is onto.

Consider now an element $b/Z^k$of degree zero. We can write b as: $b = \sum_{i=0}^{k} f_i(X, \delta) Z^{k-i}$ with degree$f_i(X, \delta) = i$.

$b/Z^k \otimes Z^\ell \longrightarrow Z^{\ell-k} \sum_{i=0}^{k} f_i(X, \delta) Z^{k-i} = 0$ in $(B_n)_Z$.

It follows there exist $t \geq 0$ such that $Z^t \sum_{i=0}^{k} f_i(X, \delta) Z^{k-i} = 0$ in B_n.

Therefore: $b/Z^k \otimes Z^\ell = bZ^t/Z^{k+t} \otimes Z^\ell = 0$.

We have proved μ is an isomorphism. □

The inclusion $K \longrightarrow K[Z, Z^{-1}]$ induces a flat morphism: $A_n \longrightarrow A_n \underset{K}{\otimes} K[Z, Z^{-1}] = A_n[Z, Z^{-1}]$ and there is a pair of adjoint functors: $A_n[Z, Z^{-1}] \otimes - : Mod_{A_n} \longrightarrow Gr_{A_n[Z,Z^{-1}]}$, $res_A : Gr_{A_n[Z,Z^{-1}]} \longrightarrow Mod_{A_n}$, where res_A is the restriction.

The following result is a particular case of a theorem given by Dade [Da]. We include the proof for completeness.

Theorem 3. *The functors* res_A*,* $A_n[Z, Z^{-1}] \otimes -$ *are exact inverse equivalences.*

Proof. It is clear both functors are exact.

Let $M = \underset{i \in Z}{\oplus} M_i$ be a graded $A_n[Z, Z^{-1}]$-module. Multiplication induces a morphism of graded modules: $\mu : A_n[Z, Z^{-1}] \underset{A}{\otimes} M_0 \to M$.

If $Z^k m = 0$, then $m = Z^{-k} Z^k m = 0$ and μ is injective, but given $m \in M_k$, $\mu(Z^k \otimes Z^{-k} m) = m$.

It follows μ is an isomorphism.

Moreover, if $f : M \longrightarrow N$ is a morphism of graded $A_n[Z, Z^{-1}]$-modules, the following diagram commutes:

$$\begin{array}{ccc} A_n[Z, Z^{-1}] \underset{A}{\otimes} M_0 & \longrightarrow & M \\ \downarrow 1 \otimes f_0 & & \downarrow f \\ A_n[Z, Z^{-1}] \underset{A}{\otimes} N_0 & \longrightarrow & N \end{array}$$

It follows, $(A_n[Z, Z^{-1}] \underset{A}{\otimes} -) res_A \cong 1$.

Given an A_n-module M, it is clear $res_A(A_n[Z, Z^{-1}] \underset{A}{\otimes} M) \cong M$ and given a morphism of A_n-modules $f : M \to N$, $res_A(1 \otimes f) = f$.

Therefore $res_A(A_n[Z, Z^{-1}] \underset{A}{\otimes} -) \cong 1$. □

Corollary 2. *The equivalences* res_A*,* $A_n[Z, Z^{-1}] \otimes -$ *preserve projective modules, irreducible modules, send left ideals to left ideals giving an order preserving bijection.*

We will study now the relations between Gr_{B_n} and $Gr_{(B_n)_Z}$. We denote by Q the localization functor $Q : Gr_{B_n} \to Gr_{(B_n)_Z}$, $M \to M_Z$, where

$M_Z = (B_n)_z \underset{B}{\otimes} M \cong K[Z, Z^{-1}] \underset{K[Z]}{\otimes} B_n \underset{B_n}{\otimes} M \cong K[Z, Z^{-1}] \underset{K[Z]}{\otimes} M.$

If we denote by gr_{B_n} and $gr_{(B_n)_Z}$ the categories of finitely generated graded B_n and $(B_n)_Z$-modules, respectively. Then Q restricts to a functor $Q : gr_{B_n} \to gr_{(B_n)_Z}$.

Definition 2. *Given a B_n-module M,define the Z-torsion of M as: $t_Z(M) = \{m \in M \mid$there exists $n > 0$ with $Z^n m = 0$ $\}$. It is clear t_Z is an idempotent radical. We say M is Z-torsion when $t_Z(M) = M$ and Z-torsion free if $t_Z(M) = 0$.*

The kernel of the natural map $M \longrightarrow M_Z$ is $t_Z(M)$.

Proposition 6. *i) Let $f : M \longrightarrow N$ be a morphism of graded B_n-modules. Then $f_Z : M_Z \longrightarrow N_Z$ is zero if and only if f factors through a Z -torsion module.*

ii) Let $\varphi : M_Z \longrightarrow N_Z$ be a morphism of finitely generated graded $(B_n)_Z$-modules, there exists an integer $k \geq 0$ and a map $f : Z^k M \longrightarrow N$ such that the composition $M_Z \longrightarrow (Z^k M)_Z \longrightarrow N_Z$ is φ and $M_Z \longrightarrow (Z^k M)_Z$ is an isomorphism of graded modules.

ii) Let M be a finitely generated graded $(B_n)_Z$-module. The there exists a finitely generated B_n-sub module $\overline{M}$ of M such that $(\overline{M})_Z \cong M_Z$.

Proof. i) Let $f : M \to N$ be a morphism such that $f_Z : M_Z \longrightarrow N_Z$, $f_Z = 0$.Let $m \in M$ with $f(m)/1 = 0$.Then there exist some $k \geq 0$ such that $Z^k f(m) = 0$. It follows $f(M)$is of Z-torsion and the map f factors as $f = j\overline{f}$ with $j : t_Z(M) \to M$ the inclusion and $\overline{f} : M \to t_Z(M)$ the restriction of f.

Conversely, $f = gh$ with $h : M \to L$ and $g : L \to M$ maps and L of Z-torsion, then $L_Z = 0$ and $f_Z = g_Z h_Z = 0$.

ii) Let $\varphi : M_Z \longrightarrow N_Z$ be a morphism of finitely generated graded $(B_n)_Z$-modules. Let m_1, m_2,... m_k be a set of homogeneous generators of M_z with degree $m_i = d_i$ and let $d = \max\{d_i\}$.Then $\varphi(m_i) = \sum_j n_{i,j} \otimes Z^{k_{i.j}}$, degree $n_{i,j} + k_{i,j} = d_i$.

If $k_{i,j} \geq 0$, then $n_{i,j} \otimes Z^{k_{i.j}} = n_{i,j} Z^{k_{i.j}} \otimes 1$,hence we may assume $\varphi(m_i) = \sum_j n_{i,j} \otimes Z^{k_{i.j}}$, with $k_{i,j} \leq 0$

Let $k =\max\{ -k_{i,j}\}$. Then $\varphi(m_i) = \sum_j n_{i,j} Z^{k+k_{i,j}} \otimes Z^{-k} = n_i \otimes Z^{-k}$, degree $n_i = d_i + k$.

Consider the restriction to $Z^k M$ of the map: $M \xrightarrow{j} M \underset{K[Z]}{\otimes} K[Z, Z^{-1}] \xrightarrow{\varphi} N \underset{K[Z]}{\otimes} K[Z, Z^{-1}]$, $f : Z^k M \longrightarrow N \otimes 1 \cong N$.The map f is a degree zero map.

We have an exact sequence: $0 \to Z^k M \to M \to M/Z^k M \to 0$, with $M/Z^k M$ of Z-torsion. Localizing, there exist an isomorphism $(Z^k M)_Z \cong M_Z$.There is a map $f_Z : (Z^k M)_Z \to N_Z$, given by $f_Z(m/Z^\ell) = f(Z^k(m/Z^{k+\ell}) = f(Z^k m)/Z^{\ell+k} = \varphi(Z^k m)/Z^{\ell+k} = Z^k \varphi(m)/Z^{\ell+k} = \varphi(m/Z^\ell)$.

iii) Let M be a finitely generated $(B_n)_Z$-module with homogeneous generators: m_1, m_2,... m_k of degree $m_i = d_i$.

By restriction, M is a B_n-module. Let $\overline{M}$ be the B_n-submodule of M generated by m_1, m_2,... m_k.

Localizing we get $\overline{M}_Z = (B_n)_Z \underset{B}{\otimes} \overline{M} \cong B_n \underset{K[Z]}{\otimes} K[Z,Z^{-1}] \underset{B}{\otimes} \overline{M} \cong K[Z,Z^{-1}] \underset{K[Z]}{\otimes} \overline{M}$.

Let $\mu : K[Z,Z^{-1}] \underset{K[Z]}{\otimes} \overline{M} \to M$ be the map given by multiplication.

The homogeneous elements of $K[Z,Z^{-1}] \underset{K[Z]}{\otimes} \overline{M}$ are of the form $Z^{-k} \otimes m$, hence $\mu(Z^{-k} \otimes m) = Z^{-k}m = 0$ implies $m = Z^k Z^{-k} m = 0$.

Let m be an element of M homogeneous of degree k. It has form: $m = \sum b_i/Z^{n_i} m_i$, degree $b_i + d_i - n_i = k$. Set $n = \max\{n_i\}$.

Then $Z^n m = \sum b_i Z^{n-n_i} m_i = \overline{m}$ is an element of $\overline{M}$ of degree $k+n$ and $\mu(Z^{-k} \otimes \overline{m}) = m$. □

Corollary 3. *Let M, N be finitely generated graded B_n-modules. A map $\varphi : M_Z \to N_Z$ is an isomorphism if and only if there exists a map $f : Z^k M \to N$ such that $Kerf$, $Cokerf$ are of Z-torsion and $f_Z = \varphi$. If there is a map $f : Z^k M \to N$ such that f_Z is an isomorphism then Kerf , $Cokerf$ are of Z-torsion even when M or N are not finitely generated.*

We shall define another torsion theory on Gr_{B_n}.

Let M be a graded B_n-module, $t(M) = \sum_{L \in J} L$ and $J = \{L$ |sub module of M $\dim_K L < \infty\}$.

Claim: $t(M/t(M)) = 0$.

Let N be a finitely generated sub module of M such that $N + t(M)/t(M) = N/N \cap t(M)$ is finite dimensional over K.Since $B_n$is noetherian is finitely generated, hence of finite dimension over K. It follows N is finite dimensional, so $N \subset t(M)$.

Let N be an arbitrary sub module of $M$with $N + t(M)/t(M)$finite dimensional over K, then $N = \sum N_i$, with N_i finitely generated, each $N_i + t(M)/t(M)$ is finite dimensional, therefore $N_i \subset t(M)$.It follows $N \subset t(M)$.

3. The derived Categories $D^b(QgrB_n)$ and $D^b(gr(B_n)_Z)$.

In this section we will study the relations between the derived categories $D^b(QgrB_n)$ and $D^b(gr(B_n)_Z)$ and their relations with the stable category $\underline{gr}_{B_n^!}$ of the shrike algebra $B_n^!$ of B_n.

Definition 3. *We say that a (graded) B_n-module is torsion if $t(M) = M$ and torsion free if $t(M) = 0$.*

It is clear $t(M)$ is Z-torsion and $t(M) \subset t_Z(M)$. Therefore if M is torsion then it is Z -torsion and if M is Z-torsion free then it is torsion free.

The torsion free modules form a Serre (or thick) subcategory of Gr_{B_n}we localize with respect to this subcategory as explained in [Ga],[P] . Denote by $QGrB_n$ the quotient category and let $\pi : Gr_{B_n} \to QGrB_n$ be the quotient functor, $QGrB_n = GrB_n/$Torsion, is an abelian category with enough injectives and π is an exact functor. When taking this quotient we are inverting the maps of B_n-graded modules, $f : M \to N$ such that Kerf and $Cokerf$ are torsion.

The category $QGrB_n$ has the same objects as GrB_n and maps:

$Hom_{QGrB_n}(\pi(M), \pi(N)) = \varinjlim Hom_{GrB_n}(M', N/t(M))$, the limit running through all the sub modules $M'$of M such that M/M' is torsion.

If M is a finitely generated module then the limit has a simpler form:

$Hom_{QGrB_n}(\pi(M),\pi(N)) = \varinjlim_k Hom_{GrB_n}(M_{\geq k}, N/t(M))$.

In case N is torsion free: $Hom_{QGrB_n}(\pi(M),\pi(N)) = \varinjlim_k Hom_{GrB_n}(M_{\geq k}, N)$.

The functor $\pi : Gr_{B_n} \to QGrB_n$ has a right adjoint: $\varpi : QGrB_n \to Gr_{B_n}$ such that $\pi\varpi \cong 1$. $[PN]$.

If we denote by gr_{B_n} the category of finitely generated graded B_n-modules and by $QgrB_n$ the full subcategory of $QGrB_n$ consisting of the objects $\pi(N)$ with N finitely generated, then the functor π induces by restriction a functor: $\pi : gr_{B_n} \to QgrB_n$.The kernel of π is: $Ker\pi = \{M \in gr_{B_n} \mid \pi(M) = 0\} = \{M \in gr_{B_n} \mid t(M) = M\}$.

In the other hand, the functor: $(B_n)_Z \underset{B}{\otimes} - : gr_{B_n} \to gr(B_n)_Z$ has kernel: $Ker((B_n)_Z \underset{B}{\otimes} -) = \{M \in gr_{B_n} \mid M_Z = 0\}$.

It follows: $Ker\pi \subset Ker((B_n)_Z \underset{B}{\otimes} -)$.

According to [P] (pag. 173 Cor. 3.11) there exists a unique functor ψ such that the following diagram commutes:

$$\begin{tikzcd} gr_{B_n} \arrow[rr, "\pi"] \arrow[dr, "(B_n)_Z \underset{B}{\otimes} -"'] & & QgrB_n \arrow[dl, "\psi"] \\ & gr(B_n)_Z & \end{tikzcd}$$

This is: $\psi\pi = (B_n)_Z \underset{B}{\otimes} -$.

Proposition 7. *The functor $\psi : QgrB_n \to gr(B_n)_Z$ is exact.*

Proof. Let $0 \to \pi(M) \xrightarrow{\hat{f}} \pi(N) \xrightarrow{\hat{g}} \pi(L) \to 0$ be an exact sequence in $QgrB_n$.We may assume M, N, L torsion free. Then::

$\hat{f} \in \varinjlim_k Hom_{GrB_n}(M_{\geq k}, N)$ and $\hat{g} \in \varinjlim_s Hom_{GrB_n}(N_{\geq s}, L)$.

There exist exact sequences: $0 \to M_{\geq k+1} \to M_{\geq k} \to M_{\geq k}/M_{\geq k+1} \to 0$ which induces an exact sequence:

$0 \to Hom_{GrB_n}(M_{\geq k}/M_{\geq k+1}, N) \to Hom_{GrB_n}(M_{\geq k}, N) \to Hom_{GrB_n}(M_{\geq k+1}, N)$.

Since we are assuming N is torsion free $Hom_{GrB_n}(M_{\geq k}/M_{\geq k+1}, N) = 0$.

Hence $\hat{f} \in Hom_{QGrB_n}(\pi(M),\pi(N)) = \bigcup_{k\geq 0} Hom_{GrB_n}(M_{\geq k}, N)$.

The map $\hat{f}$ is represented by $f : M_{\geq k} \to N$. Similarly, $\hat{g}$ is represented by a map $g : N_{\geq \ell} \to L$ and we have a sequence: : $M_{\geq k+\ell} \xrightarrow{f} N_{\geq \ell} \xrightarrow{g} L$ with $(\widehat{gf}) = \hat{g}\hat{f} = 0$, which implies gf factors through a torsion module, but L torsion free implies $gf = 0$. Since $M_{\geq k+\ell}$ is torsion free, f is a monomorphism. If $Co\ker g$ is torsion, there exists an $s \geq 0$ such that $Co\ker g_{\geq s} = 0$. Taking a large enough truncation we obtain a sequence: $M_{\geq s} \xrightarrow{f} N_{\geq s} \xrightarrow{g} L_{\geq s}$ with f a monomorphism, g an epimorphism and $gf = 0$.

Consider the exact sequences: $0 \to M_{\geq s} \xrightarrow{f''} Kerg \to H \to 0$, $0 \to Kerg \to N_{\geq s} \to L_{\geq s} \to 0$.

Applying π we obtain the following isomorphism of exact sequences:

$$\begin{array}{ccccccc} 0\to & \pi(M_{\geq s}) & \overset{\pi f''}{\to} & \pi(Kerg) & \to & \pi(H) & \to 0 \\ & \downarrow\cong & & \downarrow\cong & & & \\ 0\to & \pi(M) & \overset{\hat{f}}{\to} & Ker\hat{g} & \to & 0 & \end{array}$$

It follows $\pi(H)=0$ and H is torsion, so there exists an integer $t\geq 0$ such that $H_{\geq t}=0$.Finally taking a large enough truncation we get an exact sequence: $0\to M_{\geq s}\overset{f}{\to} N_{\geq s}\overset{g}{\to} L_{\geq s}\to 0$ such that the following sequences are isomorphic:

$$\begin{array}{ccccccc} 0\to & \pi(M_{\geq s}) & \overset{\pi f}{\to} & \pi(N_{\geq s}) & \overset{\pi g}{\to} & \pi(L_{\geq s}) & \to 0 \\ & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\ 0\to & \pi(M) & \overset{\hat{f}}{\to} & \pi(N) & \overset{\hat{g}}{\to} & \pi(L) & \to 0 \end{array}$$

Applying ψ we have an exact sequence: $0\to\psi\pi(M)\overset{\psi\hat{f}}{\to}\psi\pi(N)\overset{\psi\hat{g}}{\to}\psi\pi(L)\to 0$, which is isomorphic to $0\to(M_{\geq s})_Z\overset{f_Z}{\to}(N_{\geq s})_Z\overset{g_Z}{\to}(L_{\geq s})_Z\to 0$.

We have proved ψ is exact. □

The functor ψ has a derived functor: $D(\psi): D^b(QgrB_n)\to D^b(gr(B_n)_Z)$, we will study next its properties.

Observe $QgrB_n$ does not have neither enough projective nor enough injective objects.

Lemma 5. *Let $0\to N_1\overset{d_1}{\to}N_2\overset{d_2}{\to}...N_{\ell-1}\overset{d_{\ell-1}}{\to}N_\ell\to 0$ be a sequence of B_n-modules and assume the compositions d_id_{i-1} factors through a module of Z-torsion. Then there exists a complex: $0\to N_1\overset{\hat{d}_1}{\to}N_2\oplus t_Z(N_1)\overset{\hat{d}_2}{\to}N_3\oplus t_Z(N_2)...N_{\ell-1}$ $\oplus t_Z(N_\ell)\overset{\hat{d}_{\ell-1}}{\to}N_\ell\to 0$, where $\hat{d}_1=\begin{bmatrix}-d_1\\ s_1\end{bmatrix}$, $\hat{d}_i=\begin{bmatrix}(-1)^id_i & j_{i+1}\\ s_i & (-1)^id'_{i+1}\end{bmatrix}$ and $\hat{d}_{\ell-1}=\begin{bmatrix}-d_{\ell-1} & j_\ell\end{bmatrix}$,where the maps $j_i: t_Z(N_i)\to N_i$ are the natural inclusions.*

Proof. Each morphism $d_i: N_i\to N_{i+1}$ induces by restriction a map $d'_i: t_Z(N_i)\to t_Z(N_{i+1})$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} t_Z(N_1) & \overset{d'_1}{\to} & t_Z(N_2) & \overset{d'_2}{\to} & t_Z(N_3) & \to ... & t_Z(N_{\ell-1}) & \overset{d'_{\ell-1}}{\to} & t_Z(N_\ell) \\ \downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 & & \downarrow j_{\ell-1} & & \downarrow j_\ell \\ N_1 & \overset{d_1}{\to} & N_2 & \overset{d_2}{\to} & N_3 & & N_{\ell-1} & \overset{d_{\ell-1}}{\to} & N_\ell \end{array}$$

Since the compositions d_id_{i-1} factors through a module of Z-torsion, there exist maps $s_i: N_i\to t_Z(N_{i+2})$ such that $j_{i+2}s_i=d_{i+1}d_i$.

We have the following equalities: $j_{i+2}s_ij_i=d_{i+1}d_ij_i=d_{i+1}j_id'_i=j_{i+2}\ d'_{i+1}d'_i$ and j_{i+2} a monomorphism implies $s_ij_i=d'_{i+1}d'_i$.

We can easily check $\hat{d}_{i+1}\hat{d}_i=0$. □

Proposition 8. *Denote by Q the localization functor $Q=(B_n)_Z\underset{B}{\otimes}-$ and by $C^b(-)$, the category of bounded complexes. The induced functor $C^b(Q): C^b(gr_{B_n})\to C^b(gr_{(B_n)_Z})$ is dense.*

Proof. Let $0\to\hat{M}_1\overset{\delta_1}{\to}\hat{M}_2\overset{\delta_2}{\to}...\hat{M}_{\ell-1}\overset{\delta_{\ell-1}}{\to}\hat{M}_\ell\to 0$ be a complex in $C^b(gr_{(B_n)_Z})$.

For each $\hat{M}_i$there exists a finitely generated graded B_n-submodule M_i such that $(M_i)_Z\cong\hat{M}_i$and a graded morphism $d_i: Z^{k_i}M_i\to M_{i+1}$of B_n-modules such that

$(d_i)_Z : (Z^{k_i}M_i)_Z \to (M_{i+1})_Z$ is isomorphic $\delta_i : \hat{M}_i \to \hat{M}_{i+1}$. Let k be $k = \sum_{i=0}^{\ell} k^i$. We then have a chain of B_n-morphisms:

$Z^k M_1 \overset{d_1}{\to} Z^{k_2+..k_\ell} M_2 \overset{d_2}{\to} Z^{k_3+..k_\ell} M_3 \overset{d_3}{\to} ...Z^{k_{\ell-1}+..k_\ell} M_{\ell-1} \overset{d_{\ell-1}}{\to} Z^{k_\ell} M_\ell$. Changing notation write M_i instead of $Z^{k_i+..k_\ell} M_i$. We then have a chain of morphisms:

$M_1 \overset{d_1}{\to} M_2 \overset{d_2}{\to} ...M_{\ell-1} \overset{d_{\ell-1}}{\to} M_\ell$ such that $(M_1)_Z \overset{d_{1Z}}{\to} (M_2)_Z \overset{d_{2Z}}{\to} ...(M_{\ell-1})_Z \overset{d_{\ell-1Z}}{\to} (M_\ell)_Z$ is isomorphic to the complex: $0 \to \hat{M}_1 \overset{\delta_1}{\to} \hat{M}_2 \overset{\delta_2}{\to} ...\hat{M}_{\ell-1} \overset{\delta_{\ell-1}}{\to} \hat{M}_\ell \to 0$. This implies $(d_i d_{i-1})_Z = 0$, which means $d_i d_{i-1}$ factors through a Z-torsion module. By lemma? there exists a complex:

$0 \to M_1 \overset{\hat{d}_1}{\to} M_2 \oplus t_Z(M_1) \overset{\hat{d}_2}{\to} M_3 \oplus t_Z(M_2)...M_{\ell-1} \oplus t_Z(M_\ell) \overset{\hat{d}_{\ell-1}}{\to} M_\ell \to 0$ such that $0 \to (M_1)_Z \overset{\hat{d}_{1Z}}{\to} (M_2 \oplus t_Z(M_1))_Z \overset{\hat{d}_{2Z}}{\to} (M_3 \oplus t_Z(M_2))_Z...(M_{\ell-1} \oplus t_Z(M_\ell))_Z \overset{\hat{d}_{\ell-1Z}}{\to} (M_\ell)_Z \to 0$ is isomorphic to: $0 \to \hat{M}_1 \overset{\delta_1}{\to} \hat{M}_2 \overset{\delta_2}{\to} ...\hat{M}_{\ell-1} \overset{\delta_{\ell-1}}{\to} \hat{M}_\ell \to 0$. □

Corollary 4. *The functor $C^b(\psi) : C^b(Qgr_{B_n}) \to C^b(gr_{(B_n)_Z})$ is dense.*

Proof. There are functors $C^b(\pi) : C^b(gr_{B_n}) \to C^b(Qgr_{B_n})$ and $C^b(\psi) : C^b(Qgr_{B_n}) \to C^b(gr_{(B_n)_Z})$ such that $C^b(\psi)\ C^b(\pi) = C^b(Q)$ and $C^b(Q)$ dense implies $C^b(\psi)$ is dense. □

Corollary 5. *The induced functors $K^b(Q) : K^b(gr_{B_n}) \to K^b(gr_{(B_n)_Z})$ and $K^b(\psi) : K^b(Qgr_{B_n}) \to K^b(gr_{(B_n)_Z})$ are dense.*

Proof. Interpreting $K^b(\mathcal{A})$ as the stable category of $C^b(\mathcal{A})$, denote by $\tau : C^b(\mathcal{A}) \to K^b(\mathcal{A})$ the corresponding functor. There is a commutative diagram:

$$\begin{array}{ccc} C^b(gr_{B_n}) & \overset{C^b(Q)}{\to} & C^b(gr_{(B_n)_Z}) \\ \downarrow \tau & & \downarrow \tau \\ K^b(gr_{B_n}) & \overset{K^b(Q)}{\to} & K^b(gr_{(B_n)_Z}) \end{array}$$

Since the functors τ and $C^b(Q)$ are dense, the functor $K^b(Q)$ is dense.

As above we have isomorphisms: $K^b(\psi)\ K^b(\pi) \cong K^b(Q)$. It follows $K^b(\psi)$ is dense. □

Corollary 6. *The induced functors $D^b(Q) : D^b(gr_{B_n}) \to D^b(gr_{(B_n)_Z})$ and $D^b(\psi) : D^b(Qgr_{B_n}) \to D^b(gr_{(B_n)_Z})$ are dense.*

Proof. Since the functors $\pi : gr_{B_n} \to QgrB_n$ and $\psi : QgrB_n \to gr(B_n)_Z$ are exact they induce derived functors: $D^b(\pi) : D^b(gr_{B_n}) \to D^b(QgrB_n)$ and $D^b(\psi) : D^b(QgrB_n) \to D^b(gr(B_n)_Z)$ such that $D^b(\psi)D^b(\pi) = D^b(Q)$.

There is a commutative exact diagram:

$$\begin{array}{ccc} K^b(gr_{B_n}) & \overset{K^b(Q)}{\to} & K^b(gr_{(B_n)_Z}) \\ \downarrow & & \downarrow \\ D^b(gr_{B_n}) & \overset{D^b(Q)}{\to} & D^b(gr_{(B_n)_Z}) \end{array}$$

where the functors corresponding to the columns are dense, hence $D^b(Q)$ is dense, which in turn implies $D^b(\psi)$ is dense. □

We will describe next he kernel of the functor $D^b(\psi)$. By definition, $KerD^b(\psi) = \{\hat{M}^\circ \mid D^b(\psi)(\hat{M}^\circ) \text{ is acyclic}\}$.

Proposition 9. *There is the following description of* $\mathcal{T} = KerD^b(\psi)$. $KerD^b(\psi) = \{\pi M^\circ \mid M^\circ \in D^b(gr_{B_n})$ *such that for all* i $H^i(M^\circ)$ *is of* Z*-torsion*$\}$.

Proof. The kernel of the functor $D^b(\psi)$ is the category $\mathcal{T}$ of complexes: $\tilde{N}^\circ : 0 \to \pi N_1 \overset{\hat{d_1}}{\to} \pi N_2 \overset{\hat{d_2}}{\to} \pi N_3 ... \pi N_{\ell-1} \overset{\hat{d_{\ell-1}}}{\to} \pi N_\ell \to 0$, such that:

$0 \to \psi\pi N_1 \overset{\psi\hat{d_1}}{\to} \psi\pi N_2 \overset{\psi\hat{d_2}}{\to} \psi\pi N_3 ... \psi\pi N_{\ell-1} \overset{\psi\hat{d_{\ell-1}}}{\to} \psi\pi N_\ell \to 0$ is acyclic.

Proceeding as above, we may assume each N_i and each map $\hat{d_i}$ lifts to a map $d_i : (N_i)_{\geq k} \to N_{i+1}$ such that the map $\pi(d_i) : \pi((N_i)_{\geq k}) \to \pi(N_{i+1})$ is isomorphic to $\hat{d_i} : \pi N_i \to \pi N_{i+1}$.

Taking a large enough truncation we get a complex of B_n-modules: $N^\circ_{\geq k} : 0 \to (N_1)_{\geq k} \overset{d_1}{\to} (N_2)_{\geq k} \overset{d_2}{\to} ...(N_{\ell-1})_{\geq k} \overset{d_{\ell-1}}{\to} (N_\ell)_{\geq k} \to 0$ such that $\pi(N^\circ_{\geq k}) \cong \tilde{N}^\circ$.

The complex: $(N^\circ_{\geq k})_Z : 0 \to ((N_1)_{\geq k})_Z \overset{d_{1Z}}{\to} ((N_2)_{\geq k})_Z \overset{d_{2Z}}{\to} ...((N_{\ell-1})_{\geq k})_Z \overset{d_{\ell-1Z}}{\to} ((N_\ell)_{\geq k})_Z \to 0$ is isomorphic to $0 \to \psi\pi N_1 \overset{\psi\hat{d_1}}{\to} \psi\pi N_2 \overset{\psi\hat{d_2}}{\to} \psi\pi N_3 ... \psi\pi N_{\ell-1} \overset{\psi\hat{d_{\ell-1}}}{\to} \psi\pi N_\ell \to 0$, hence it is acyclic.

Changing notation we have a complex of B_n-modules: $N^\circ : 0 \to N_1 \overset{d_1}{\to} N_2 \overset{d_2}{\to} ...N_{\ell-1} \overset{d_{\ell-1}}{\to} N_\ell \to 0$ such that $(N^\circ)_Z : 0 \to (N_1)_Z \overset{d_{1Z}}{\to} (N_2)_Z \overset{d_{2Z}}{\to} ...(N_{\ell-1})_Z \overset{d_{\ell-1Z}}{\to} (N_\ell)_Z \to 0$ is acyclic.

We have exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 \to & Kerd_1 & \to & N_1 & \overset{d_1}{\to} & \operatorname{Im} d_1 & \to 0 & & \\
 & & & & & \downarrow j & & & \\
 & & & 0 \to & & Kerd_2 & \to & N_2 \overset{d_2}{\to} & N_3 \\
 & & & & & \downarrow & & & \\
 & & & & & H^1(N^\circ) & & & \\
 & & & & & \downarrow & & & \\
 & & & & & 0 & & &
\end{array}$$

Localizing we get exact sequences:

$$\begin{array}{ccccccccc}
 & & & & & 0 & & & \\
 & & & & & \downarrow & & & \\
0 \to & (Kerd_1)_Z & \to & (N_1)_Z & \overset{d_{1Z}}{\to} & (\operatorname{Im} d_1)_Z & \to 0 & & \\
 & & & & & \downarrow j_Z & & & \\
 & & & & 0 \to & (Kerd_2)_Z & \to & (N_2)_Z \overset{d_{2Z}}{\to} & (N_3)_Z \\
 & & & & & \downarrow & & & \\
 & & & & & H^1(N^\circ)_Z & & & \\
 & & & & & \downarrow & & & \\
 & & & & & 0 & & &
\end{array}$$

where j_Z, d_{1Z} are isomorphisms, hence $H^0(N^\circ)_Z = 0$, $H^1(N^\circ)_Z = 0$.Therefore $H^0(N^\circ)$ and $H^1(N^\circ)$ are Z-torsion. More generally for each i the modules $H^i(N^\circ)$ are Z-torsion. □

Corollary 7. *The Nakayama automorphism* $\sigma : B_n \to B_n$ *induces an autoequivalence* $D^b(\sigma) : D^b(gr_{B_n}) \to D^b(gr_{B_n})$ *and* $\mathcal{T}$ *is invariant under* $D^b(\sigma)$.

Proof. We saw in Section one that given an automorphism of graded algebras $\sigma : B_n \to B_n$, there is an autoequivalence $gr_{B_n} \to gr_{B_n}$, that we also denote by σ, such that $\sigma(M)$ is the module M with twisted multiplication $b \in B_n$ and $m \in M$, $b * m = \sigma(b)m$, clearly σ is an exact functor that sends modules of finite length into modules of finite length. Then σ induces an exact functor: $\sigma : QgrB_n \to QgrB_n$.Therefor: an autoequivalence: $D^b(\sigma) : D^b(gr_{B_n}) \to D^b(gr_{B_n})$. If M is a module of Z-torsion, then σM is of Z-torsion. From this it is clear that $D^b(\sigma)$ sends an element of $\mathcal{T}$ to an element of $\mathcal{T}$. □

The category $\mathcal{T} = KerD(\psi)$ is "epasse" (thick) and we can take the Verdier quotient $D^b(QgrB_n)\ /\mathcal{T}$.

Our aim is to prove the main result of the section:

Theorem 4. *There exists an equivalence of triangulated categories:* $D^b(QgrB_n)/\mathcal{T} \cong D^b(gr(B_n)_Z)$.

Let $\hat{s}^{-1}\hat{f} : \hat{X}^{\circ} \to \hat{Y}^{\circ}$ be a map in $\Psi(\mathcal{T})$ (in Miyachi's notation). This is a roof:

$$\begin{array}{ccccc} & & \hat{K}^{\circ} & & \\ & \hat{f}\nearrow & & \nwarrow \hat{s} & \\ \hat{X}^{\circ} & & & & \hat{Y}^{\circ} \end{array}$$

, where $\hat{X}^{\circ}$ is a complex of the form: $\hat{X}^{\circ :} : 0 \to \pi X^{n_0} \xrightarrow{\hat{d}} \pi X^{n_0+1} \xrightarrow{\hat{d}} ... \to \pi X^{n_0+\ell-1} \xrightarrow{\hat{d}} \pi X^{n_0+\ell} \to 0.$

After a proper truncation there exists complexes of graded B_n-modules X°, K°, Y° such that $\pi X^{\circ} \cong \hat{X}^{\circ}$, $\pi K^{\circ} \cong \hat{K}^{\circ :}$,$\pi\ Y^{\circ} \cong \hat{Y}^{\circ :}$ and graded maps $f : X^{\circ} \to K^{\circ}$, $s : Y^{\circ} \to K^{\circ}$ such that $\pi f = \hat{f}$, $\pi s = \hat{s}$, the roof $\hat{s}^{-1}\hat{f}$ becomes:

$$\begin{array}{ccccc} & & \pi K^{\circ} & & \\ & \pi f\nearrow & & \nwarrow \pi s & \\ \pi X^{\circ} & & & & \pi Y^{\circ} \end{array}$$

, where πs is a quasi isomorphism.

There is a triangle in $K^b(grB_n) : X^{\circ} \xrightarrow{f} K^{\circ} \xrightarrow{g} Z^{\circ} \xrightarrow{h} X^{\circ}[-1]$ which induces a morphism of triangles:

$$\begin{array}{ccccccc} X^{\circ} & \xrightarrow{f} & K^{\circ} & \xrightarrow{g} & Z^{\circ} & \xrightarrow{h} & X^{\circ}[-1] \\ \uparrow u & & \uparrow s & & \uparrow 1 & & \uparrow u[-1] \\ X'^{\circ} & \to & Y^{\circ} & \xrightarrow{gs} & Z^{\circ} & \to & X'^{\circ}[-1] \end{array}$$

Applying π we obtain a morphism of triangles:

$$\begin{array}{ccccccc} \pi X^{\circ} & \xrightarrow{\pi f} & \pi K^{\circ} & \xrightarrow{\pi g} & \pi Z^{\circ} & \pi \xrightarrow{h} & \pi X^{\circ}[-1] \\ \uparrow \pi u & & \uparrow \pi s & & \uparrow 1 & & \uparrow \pi u[-1] \\ \pi X'^{\circ} & \to & \pi Y^{\circ} & \xrightarrow{\pi gs} & \pi Z^{\circ} & \to & \pi X'^{\circ}[-1] \end{array}$$

By definition of $\Psi(\mathcal{T})$ the object $\pi Z^{\circ} \in \mathcal{T}$,which means Z° has homology of Z-torsion. The maps πs, πu are quasi isomorphisms. Applying the functor ψ we obtain a triangle: $\psi\pi X^{\circ} \xrightarrow{\psi\pi f} \psi\pi K^{\circ} \xrightarrow{\psi\pi g} \psi\pi Z^{\circ} \xrightarrow{\psi\pi h} \psi\pi X^{\circ}[-1]$ where $\psi\pi Z^{\circ}$is acyclic. It follows $\psi\pi f$ is invertible in $D^b(gr(B_n)_Z)$.

We have proved the functor: $D^b(\psi) : D^b(Qgr_{B_n}) \to D^b(gr_{(B_n)_Z})$ sends elements of $\Psi(\mathcal{T})$ to invertible elements in $D^b(gr(B_n)_Z)$.By $[Mi]$ Prop. 712, there exists a functor $\theta : D^b(Qgr_{B_n})/\mathcal{T} \to D^b(gr_{(B_n)_Z})$ such that the triangle:

$$\begin{array}{ccccc} D^b(Qgr_{B_n}) & & \overset{D^b(\psi)}{\to} & & D^b(gr(B_n)_Z) \\ & Q \searrow & & \nearrow \theta & \\ & & D^b(QgrB_n)/\mathcal{T} & & \end{array}$$, commutes.

Since $D^b(\psi)$is dense, so is θ.

Before proving θ is an equivalence, we will need two lemmas:

Lemma 6. *Let K°, L° be complexes in $C^b(gr_{B_n})$ and let $\hat{f} : K^\circ_Z \to L^\circ_Z$ be a morphism of complexes of graded $(B_n)_Z$-modules. Then there exists a bounded complex of graded B_n-modules N° and a map of complexes $f : N^\circ \to L^\circ$ such that $N^\circ_Z \cong K^\circ_Z$ and $f_Z = \hat{f}$.*

Proof. Let K°, L° be the complexes: K°: $0 \to K^0 \overset{d}{\to} K^1 \overset{d}{\to} ...K^{n-1} \overset{d}{\to} K^n \to 0$ and $L^\circ : 0 \to L^0 \overset{d}{\to} L^1 \overset{d}{\to} ...L^{n-1} \overset{d}{\to} L^n \to 0$.

Each map $\hat{f}_i : K^i_Z \to L^i_Z$ lifts to a map $f_i : Z^{k_i}K^i \to L^i$ such that $(f_i)_Z = \hat{f}_i$. Let k be $k = \max\{k_j\}$. Then we have the following diagram:

$$\begin{array}{ccccccccc} 0 \to & Z^kK^0 & \overset{d}{\to} & Z^kK^1 & \overset{d}{\to} & Z^kK^2 & \overset{d}{\to} ... & Z^kK^n & \to 0 \\ & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_n & \\ 0 \to & L^0 & \overset{d}{\to} & L^1 & \overset{d}{\to} & L^2 & \overset{d}{\to} ... & L^n & \to 0 \end{array}$$

where $(df_{i-1} - f_i d)_Z = d\hat{f}_{i-1} - \hat{f}_i d = 0$.The map $df_{i-1} - f_i d$ factors though $t_Z(L^i)$.

There exist maps $s_{i-1} : Z^kK^{i-1} \to t_Z(L^i)$ and $j_i : t_Z(L^i) \to L^i$ such that $f_i d - df_{i-1} = j_i s_{i-1}$ and the diagrams:

$$\begin{array}{ccc} t_Z(L^{i-1}) & \overset{d'}{\to} & t_Z(L^i) \\ \downarrow j_{i-1} & & \downarrow j_i \\ L^{i-1} & \overset{d}{\to} & L^i \end{array}$$ commute.

We have the following equalities: $(f_i d - df_{i-1})d = j_i s_{i-1} d$, $-df_{i-1}d = j_i s_{i-1} d$ and $d(f_{i-1}d - df_{i-2}) = df_{i-1}f = dj_{i-1}s_{i-2} = j_i ds_{i-2}$.

But j_i mono implies $s_{i-1}d + ds_{i-2} = 0$.

We have proved that:

$N^\circ : 0 \to Z^kK^0 \overset{\overset{\wedge}{d_0}}{\to} Z^kK^1 \oplus t_Z(L^1) \overset{\overset{\wedge}{d_1}}{\to} Z^kK^2 \oplus t_Z(L^2)... Z^kK^{n-1} \oplus t_Z(L^{n-1}) \overset{\overset{\wedge}{d_{\ell-1}}}{\to}$ $Z^kK^n \to 0$, where the maps have the following form: $\hat{d}_0 = \begin{bmatrix} d \\ s_0 \end{bmatrix}$, $\hat{d}_i = \begin{bmatrix} d & 0 \\ s_i & d' \end{bmatrix}$, $\hat{d}_{\ell-1} = \begin{bmatrix} d & 0 \end{bmatrix}$ is a complex of B_n-modules and $(f_i, -j_i) : Z^kK^i \oplus t_Z(L^i) \to L^i$, $(f, -j) : N^\circ \to L^\circ$ is a map of complexes such that $N^\circ_Z \cong K^\circ_Z$ and $(f, j)_Z = \hat{f}$. $\square$

Lemma 7. *Let K°, L° be complexes in $C^b(gr_{B_n})$ and $\hat{f} : L^\circ_Z \to K^\circ_Z$ be a morphism of complexes of graded $(B_n)_Z$-modules which is homotopic to zero. Then there exist bounded complexes of B_n-modules, M°, $N^\circ$and a map of complexes $f : N^\circ \to M^\circ$, such that f is homotopic to zero, $N^\circ_Z \cong L^\circ_Z$, $M^\circ_Z \cong K^\circ_Z$ and $f_Z \cong \hat{f}$.*

Proof. Consider the following diagram:

$$\begin{array}{cccccccccc} 0\to & L_Z^0 & \overset{d_Z}{\to} & L_Z^1 & \overset{d_Z}{\to} & L_Z^2 & \overset{d_Z}{\to}\ldots & L_Z^m & \to 0 \\ & \downarrow \hat{f}_0 & s_1\swarrow & \downarrow \hat{f}_1 & s_2\swarrow & \downarrow \hat{f}_2 & s_m\swarrow & \downarrow \hat{f}_m & \\ 0\to & K_Z^0 & \overset{d_Z}{\to} & K_Z^1 & \overset{d_Z}{\to} & K_Z^2 & \overset{d_Z}{\to}\ldots & L_Z^n & \to 0 \end{array}$$

, where $s : L_Z^\circ \to K_Z^\circ[-1]$ is the homotopy, hence $\hat{f}_i = d_Z s_i + s_{i+1}d_Z$.

For each i there exist integers k_i, k_i' and maps $t_i : Z^{k_i}L^i \to K^{i-1}$ and $f_i' : Z^{k_i'}L^i \to K^i$ such that $(t_i)_Z = s_i$ and $(f_i')_Z = \hat{f}_i$. Taking $k = \max\{k_j\}$we have maps:

$$\begin{array}{cccccccccc} 0\to & Z^kL^0 & \overset{d}{\to} & Z^kL^1 & \overset{d}{\to} & Z^kL^2 & \overset{d}{\to}\ldots & Z^kL^m & \to 0 \\ & \downarrow f_0' & t_1\swarrow & \downarrow f_1' & t_2\swarrow & \downarrow f_2' & t_m\swarrow & \downarrow f_m' & \\ 0\to & K^0 & \overset{d}{\to} & K^1 & \overset{d}{\to} & K^2 & \overset{d}{\to}\ldots & K^m & \to 0 \end{array}$$

Consider the map: $(f_i' - (t_{i+1}d + dt_i))_Z = \hat{f}_i - (s_{i+1}d_Z + d_Z s_i = 0$.As above, $f_i' - (t_{i+1}d + dt_i)$ factors through a Z-torsion module and there exist maps: $v_i : Z^kL^i \to t_Z(K^i)$ and inclusions $j_i : t_Z(K^i) \to K^i$ such that $f_i' - (t_{i+1}d + dt_i) = -j_iv_i$ or $f_i' + j_iv_i = t_{i+1}d + dt_i$.

Set $f_i = f_i' + j_iv_i$.Then $(f_i)_Z = (f_i')_Z = \hat{f}_i$.

But we have now $f_i = t_{i+1}d+dt_i$, $f_{i-1} = t_id+dt_{i-1}$ imply $f_id = dt_id = f_{i-1}d$. □

We can prove now the theorem.

i) θ is faithful.

Let $\mathcal{T}_1$ be $\mathcal{T}_1 = \{X^\circ \in C^b(gr_{B_n}) \mid H^i(X^\circ)$ is torsion for all $i\}$and $\mathcal{T}_2 = \{X^\circ \in C^b(gr_{B_n}) \mid H^i(X^\circ)$is Z-torsion for all $i\}$.

A map in $D^b(QgrB_n)/\mathcal{T}$ can be written as follows:

$$\begin{array}{ccccccccc} & & \pi K^\circ & & & & \pi L^\circ & & \\ & \pi f \nearrow & & \nwarrow \pi s & & \pi t \nearrow & & \nwarrow \pi g & \\ \pi X^\circ & & & & \pi Y^\circ & & & & \pi Z^\circ \end{array}$$

where t, $s \in \Psi(\mathcal{T}_1)$ and $g \in \Psi(\mathcal{T}_2)$.

In $K^b(gr_{B_n})$ we have maps:

$$\begin{array}{ccccccccc} & & K^\circ & & & & L^\circ & & \\ & f \nearrow & & \nwarrow s & & t \nearrow & & \nwarrow g & \\ X^\circ & & & & Y^\circ & & & & Z^\circ \end{array}$$

We have an exact sequence of complexes:

$$0 \to \quad Y^\circ \quad \overset{\mu}{\to} \quad K^\circ \oplus L^\circ \oplus I^\circ \quad \overset{\upsilon}{\to} \quad W^\circ \quad \to 0$$

Where I° is a complex which is a sum of complexes of the form: $0 \to X \overset{1}{\to} X \to 0$, hence acyclic. The maps μ, υ are of the form: $\mu = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$ and $\upsilon = \begin{bmatrix} t' & s' & \upsilon \end{bmatrix}$.

By the long homology sequence, there is an exact sequence: *)

$$\ldots\to H^{i+1}(W^\circ) \to H^i(Y^\circ) \overset{H^i(\mu)}{\to} H^i(K^\circ) \oplus H^i(L^\circ) \overset{H^i(\upsilon)}{\to} H^i(W^\circ) \to H^{i-1}(Y^\circ) \to \ldots$$

Since π is an exact functor, for any complex $\pi H^i(X^\circ) \cong H^i(\pi X^\circ)$ and the exact sequence *) induces an exact sequence: **)

$$\ldots\to \pi H^{i+1}(W^\circ) \to \pi H^i(Y^\circ) \overset{\pi H^i(\mu)}{\to} \pi H^i(K^\circ) \oplus \pi H^i(L^\circ) \overset{\pi H^i(\upsilon)}{\to} \pi H^i(W^\circ) \to \pi H^{i-1}(Y^\circ) \to \ldots$$

Which is isomorphic to the complex:

$$\ldots \to H^{i+1}(\pi W^\circ) \to H^i(\pi Y^\circ) \overset{H^i(\pi\mu)}{\to} H^i(\pi K^\circ) \oplus H^i(\pi L^\circ) \overset{H^i(\pi\upsilon)}{\to} H^i(\pi W^\circ) \to H^{i-1}(\pi Y^\circ) \to \ldots$$

The maps $H^i(\pi s), H^i(\pi t)$ are isomorphisms. Hence it follows $H^i(\pi\mu)$ is for each i a splittable monomorphism and for each i there is an exact sequence:

$$0 \to H^i(\pi Y^\circ) \overset{H^i(\pi\mu)}{\to} H^i(\pi K^\circ) \oplus H^i(\pi L^\circ) \overset{H^i(\pi\upsilon)}{\to} H^i(\pi W^\circ) \to 0$$

which can be embedded in a commutative exact diagram:

$$\begin{array}{cccccccc}
 & & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \\
 & 0 & & H^i(\pi L^\circ) & \overset{1}{\to} & H^i(\pi L^\circ) & & \\
 & \downarrow & & \downarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & \downarrow H^i(\pi s') & & \\
0 \to & H^i(\pi Y^\circ) & \to & H^i(\pi K^\circ) \oplus H^i(\pi L^\circ) & \to & H^i(\pi W^\circ) & \to 0 \\
 & \downarrow H^i(\pi s) & & \downarrow \begin{bmatrix} 1 & 0 \end{bmatrix} & & \downarrow & & \\
0 \to & H^i(\pi K^\circ) & \overset{1}{\to} & H^i(\pi K^\circ) & \to & 0 & & \\
 & \downarrow & & \downarrow & & & & \\
 & 0 & & 0 & & & &
\end{array}$$

By this and a similar diagram it follows $H^i(\pi s'), H^i(\pi t')$ are isomorphisms.

We have a commutative diagram in $K^b(Qgr_{B_n})$:

$$\begin{array}{ccccccccc}
 & & & & \pi W^\circ & & & & \\
 & & & \pi t' \nearrow & & \nwarrow \pi s' & & & \\
 & & \pi K^\circ & & & & \pi L^\circ & & \\
 & \pi f \nearrow & & \nwarrow \pi s & & \pi t \nearrow & & \nwarrow \pi g & \\
\pi X^\circ & & & & \pi Y^\circ & & & & \pi Z^\circ
\end{array}$$

Then $\theta(\ (\pi g)^{-1}\pi t(\pi s)^{-1}\pi f) = D^b(\psi)((\pi s'\pi g)^{-1}\pi t'\pi f)) = (s'_Z g_Z)^{-1} t'_Z f_Z = 0$.

But s'_Z, g_Z, t'_Z are isomorphisms in $D^b(grB_z)$. It follows $f_Z = 0$ in $D^b(grB_z)$.

Then there is a quasi isomorphism of complexes $\upsilon : \hat{N}{}^\circ \to X_Z^\circ$ such that $f_Z\upsilon$ is homotopic to zero. By Lemma ?, there is a bounded complex N° of B_n-modules and a map $\nu : N^\circ \to X^\circ$such that $N_Z^\circ \cong \hat{N}{}^\circ$ and ν_Z can be identified with υ.

According to Lemma ??. there is an integer $k \geq 0$ such that the composition of maps $Z^k N^\circ \overset{res\nu}{\to} X^\circ \overset{f}{\to} K^\circ$ is homotopic to zero and $(res\nu)_Z = \nu_Z$ is a quasi isomorphism. This implies $res\nu \in \Psi(\mathcal{T}_2)$ and $\pi f = 0$ in $D^b(QgrB_n)/\mathcal{T}$.

Therefore $\pi g)^{-1}\pi t(\pi s)^{-1}\pi f = 0$ in $D^b(QgrB_n)/\mathcal{T}$.

ii) θ is full.

Let $\begin{array}{ccc} & K_Z^\circ & \\ \hat{s} \swarrow & & \searrow \hat{f} \\ X_Z^\circ & & Y_Z^\circ \end{array}$ be a map in $D^b(gr_{(B_n)_Z})$. By lemma ?, there exists a complex:

$N^\circ : 0 \to Z^k K^0 \overset{\overset{\wedge}{d_0}}{\to} Z^k K^1 \oplus t_Z(Y^1) \overset{\overset{\wedge}{d_1}}{\to} Z^k K^2 \oplus t_Z(Y^2) \ldots Z^k K^{n-1} \oplus t_Z(Y^{n-1}) \overset{\hat{d}_{\ell-1}}{\to}$ $Z^k K^n \to 0$, where the maps have the following form: $\overset{\wedge}{d_0} = \begin{bmatrix} d \\ s_0 \end{bmatrix}$, $\overset{\wedge}{d_i} = \begin{bmatrix} d & 0 \\ s_i & d' \end{bmatrix}$,

$\hat{d}_{\ell-1} = \begin{bmatrix} d & 0 \end{bmatrix}$ and a map $f : N^\circ \to Y^\circ$ such that $N_Z^\circ \cong K_Z^\circ$ and $f_Z = \hat{f}$. Changing N° for K° we may assume $\hat{f}$ is a localized map f_Z and get a roof:

$$\begin{array}{ccccc} & & N_Z^\circ & & \\ & \hat{s} \swarrow & & \searrow f_Z & \\ X_Z^\circ & & & & Y_Z^\circ \end{array}.$$

We now lift $\hat{s}$ to a map of complexes $s : \hat{N}^\circ \to X^\circ$:

$$\begin{array}{cccccccccc} 0 \to & Z^k N^0 & \overset{\hat{d}_0}{\to} & Z^k N^1 \oplus t_Z(X^1) & \overset{\hat{d}_1}{\to} & Z^k N^2 \oplus t_Z(X^2) & \ldots & \overset{\hat{d}_{m-1}}{\to} & Z^k N^m & \to 0 \\ s: & \downarrow s_0 & & \downarrow s_1 & & \downarrow s_2 & & & \downarrow s_m & \\ 0 \to & X^0 & \overset{d}{\to} & X^1 & \overset{d}{\to} & X^2 & \ldots & \overset{d}{\to} & X^m & \to 0 \end{array} \quad s$$

with $s_z = \hat{s}$.

We have a commutative diagram:

$$\begin{array}{cccccccccc} 0 \to & Z^k N^0 & \overset{\hat{d}_0}{\to} & Z^k N^1 \oplus t_Z(X^1) & \overset{\hat{d}_1}{\to} & Z^k N^2 \oplus t_Z(X^2) & \ldots & \overset{\hat{d}_{m-1}}{\to} & Z^k N^m & \to 0 \\ (10) & \downarrow 1 & & \downarrow (10) & & \downarrow (10) & & & \downarrow 1 & \\ 0 \to & Z^k N^0 & \overset{d}{\to} & Z^k N^1 & \overset{d}{\to} & Z^k N^2 & \ldots & \overset{d}{\to} & Z^k N^m & \to 0 \end{array}$$

We obtain the following roof:

$$\begin{array}{ccccccc} & & \hat{N}^\circ & & & & \\ & s \swarrow & & \searrow (10) & & & \\ X^\circ & & & & Z^k N^\circ & & \\ & & & & & \searrow f & \\ & & & & & & Y^\circ \end{array}$$

Localizing we obtain:

$$\begin{array}{ccccccc} & & \hat{N}_Z^\circ & & & & \\ & s_Z \swarrow & & \searrow (10)_Z & & & \\ X_Z^\circ & & & & Z^k N_Z^\circ & & \\ & & & & & \searrow f_Z & \\ & & & & & & Y_Z^\circ \end{array}$$

with $\hat{N}_Z^\circ \overset{(10)_Z}{\to} Z^k N_Z^\circ \cong N_Z^\circ$ isomorphisms, $s_Z = \hat{s}$, $f_z = \hat{f}$.

We have proved θ is full.

4. The category of $\mathcal{T}$-local objects.

Let $\mathcal{F}$ be the full subcategory of $D^b(Qgr_{B_n})$ consisting of $\mathcal{T}$-local objects, this is: $\mathcal{F} = \{X^\circ \in D^b(Qgr_{B_n}) \mid Hom_{D^b(Qgr_{B_n})}(\mathcal{T}, X^\circ) = 0\}$

According to $[Mi]$, Prop. 9.8, for each $Y^\circ \in D^b(Qgr_{B_n})$ and $X^\circ \in \mathcal{F}$, $Hom_{D^b(Qgr_{B_n})}(Y^\circ, X^\circ) = Hom_{D^b(Qgr_{B_n})/\mathcal{T}}(QY^\circ, QX^\circ) \cong$
$Hom_{D^b(gr_{(B_n)_Z})}(\psi Y^\circ, \psi X^\circ)$.

In particular there is a full embedding of $\mathcal{F}$ in $D^b(gr_{(B_n)_Z})$.

According to $[MM]$ and $[MS]$ there is a duality of triangulated categories $\overline{\phi} : \underline{gr}_{B_n^!} \to D^b(Qgr_{B_n^{op}})$ induced by the duality $\phi : gr_{B_n^!} \to \mathcal{LCP}_{B_n}$, with $\mathcal{LCP}_{B_n}$ the

category of linear complexes of graded projective B_n-modules. If $M = \underset{i\geq k_0}{\oplus} M_i$ is a graded $B_n^!-$module, then $\phi(M)$ is a complex of the form:

$D(M)\otimes B_n :\to ...D(M_{k_0+n}) \underset{B_0}{\otimes} B_n[-k_0-n] \to D(M_{k_0+n-1}) \underset{B_0}{\otimes} B_n[-k_0-n+1] \to ...D(M_{k_0+1} \underset{B_0}{\otimes} B_n)[-k_0-1] \to D(M_{k_0}) \underset{B_0}{\otimes} B_n[-k_0] \to 0.$

$\overline{\phi}(M)$ is the complex:

$:\to ...\pi(D(M_{k_0+n}) \underset{B_0}{\otimes} B_n)[-k_0-n] \to \pi(D(M_{k_0+n-1}) \underset{B_0}{\otimes} B_n)[-k_0-n+1] \to ...\pi(D(M_{k_0+1}) \underset{B_0}{\otimes} B_n)[-k_0-1] \to \pi(D(M_{k_0}) \underset{B_0}{\otimes} B_n)[-k_0] \to 0$

If we compose with the usual duality we obtain an equivalence of triangulated categories: $\overline{\phi}D : \underline{gr}_{B_n^!} \to D^b(Qgr_{B_n})$.

Under the duality there is a pair $(\mathcal{F}', \mathcal{T}')$ such that $\mathcal{F}' \to \mathcal{F}$ and $\mathcal{T}' \to \mathcal{T}$ corresponds to the pair: $(\mathcal{T}, \mathcal{F})$.

We want to characterize the subcategories $\mathcal{F}', \mathcal{T}'$of $\underline{gr}_{B_n^!}$.

We shall start by recalling some properties of the finitely generated graded B_n-modules.

The algebra B_n is a Koszul algebra of finite global dimension, under such conditions, for any finitely generated graded B_n-module M there is a truncation $M_{\geq k}$ such that $M_{\geq k}[k]$ is Koszul [M]. But in Qgr_{B_n} the objects πM and $\pi M_{\geq k}$ are isomorphic, hence we can consider only Koszul B_n-modules and their shifts. Assume M is finitely generated but of infinite dimension over K. The torsion part $t(M)$ is finite dimensional over K, hence there is a torsion free truncation $M_{\geq k}$ of M, so we may assume M torsion free and Koszul.

Let´s suppose M is of Z-torsion.There exists an integer n such that $Z^{n-1}M \neq 0$ and $Z^n M = 0$. There is a filtration of M: $M \supset ZM \supset Z^2M... \supset Z^{\,n-1}M \supset 0$. Since Z is an element of degree one $(ZM)_i = ZM_{i-1}$, which implies $(Z^jM)_{\geq k} = Z^j(M_{\,\geq k-j})$.

Truncation of Koszul is Koszul and we can take large enough truncation in order to have $(Z^jM)_{\geq k}$ Koszul for all j.Changing M for $M_{\geq k}$ we may assume all Z^jM are Koszul.

There is a commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & 0\to & \Omega(M) & \to & \Omega(M/ZM) & \to & ZM \to 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & P & \xrightarrow{1} & P & & \\
 & & & & \downarrow & & \downarrow & & \\
0\to & ZM & \to & & M & \to & M/ZM & \to 0 & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

the modules $\Omega(M)$, ZM are Koszul generated in the same degree, it follows M/ZM is Koszul and for any integer $k \geq 1$ there is an exact sequence:

$0 \to \Omega^k(M) \to \Omega^k(M/ZM) \to \Omega^{k-1}(ZM) \to 0$ .By [GM1] there is an exact sequence:

$0 \to Hom_{B_n}(\Omega^{k-1}(ZM), B_{n0}) \to Hom_{B_n}(\Omega^k(M/ZM), B_{n0}) \to$
$Hom_{B_n}(\Omega^k(M), B_{n0}) \to 0$ or an exact sequence:

*) $0 \to Ext^{k-1}_{B_n}(ZM, B_{n0}) \to Ext^k_{B_n}(M/ZM, B_{n0}) \to Ext^k_{B_n}(M, B_{n0}) \to 0.$

We will denote by $F_{B_n}(N) = \underset{k\geq 0}{\oplus} Ext^k_{B_n}(N, B_{n0})$ the Koszul duality functor $F_{B_n} : K_{B_n} \to K_{B_n^!}$.

Adding all sequences *) we obtain an exact sequence:

$0 \to F_{B_n}(ZM)[-1] \to F_{B_n}(M/ZM) \to F_{B_n}(M) \to 0$

We can apply the same argument to any module Z^jM to get an exact sequence:

$0 \to F_{B_n}(Z^{j+1}M)[-j-1] \to F_{B_n}(Z^jM/Z^{j+1}M)[-j] \to F_{B_n}(Z^jM)[-j] \to 0.$

Gluing all short exact sequences we obtain a long exact sequence of Koszul up to shifting $B_n^!$-modules:

**) $0 \to F_{B_n}(Z^{n-1}M)[-n+1] \to F_{B_n}(Z^{n-2}M/Z^{n-1}M)[-n+2] ... \to$
$F_{B_n}(M/ZM) \to F_{B_n}(M) \to 0$

It will be enough to study non semismple Koszul B_n-modules N such that $ZN = 0$. They can be considered as C_n-modules.

We have the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & ZB_n^{n_0} & \xrightarrow{1} & ZB_n^{n_0} & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & \Omega_B(N) & \to & B_n^{n_0} & \to & N & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow 1 & \\
0 \to & \Omega_C(N) & \to & C_n^{n_0} & \to & N & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}$$

The algebra B_n is an integral domain and in consequence the free B_n-modules are torsion free and $ZB_n^{n_0}$is isomorphic to $B_n^{n_0}[-1]$.

The exact sequence: $0 \to ZB_n^{n_0} \to \Omega_B(N) \to \Omega_C(N) \to 0$ consists of graded modules generated in degree one and the first two term are Koszul, by $[GM]$ this implies $\Omega_C(N)$ is Koszul as B_n-module.

There is a commutative exact diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & 0 & \to & B_n^{n_0}[-1] & \to & ZB_n^{n_0} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Omega_B^2(N) & \to & B_n^{n_0+n_1}[-1] & \to & \Omega_B(N) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Omega_B\Omega_C(N) & \to & B_n^{n_1}[-1] & \to & \Omega_C(N) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

In particular $\Omega_B^2(N) \cong \Omega_B\Omega_C(N)$.

Since $\Omega_C(N) \subset C_n^{n_0}$it is a C_n-module and we have the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & ZB_n^{n_1}[-1] & \xrightarrow{1} & ZB_n^{n_1}[-1] & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & \Omega_B\Omega_C(N) & \to & B_n^{n_1}[-1] & \to & \Omega_C(N) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow 1 & \\
0 \to & \Omega_C^2(N) & \to & C_n^{n_1} & \to & \Omega_C(N) & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}$$

and an exact sequence: $0 \to B_n^{n_1}[-2] \to \Omega_B^2(N) \to \Omega_C^2(N) \to 0.$

In general there exist exact sequences:

$0 \to B_n^{n_{k-1}}[-k] \to \Omega_B^k(N) \to \Omega_C^k(N) \to 0.$

which induce exact sequences:

$0 \to \mathrm{Hom}_{B_n}(\Omega_C^k(\mathrm{N}),\mathrm{B}_{n0}) \to \mathrm{Hom}_{B_n}(\Omega_B^k(\mathrm{N}),\mathrm{B}_{n0}) \to \mathrm{Hom}_{B_n}(\mathrm{B}_n^{n_{k-1}}[\text{-k}],\mathrm{B}_{n0}) \to 0.$

The module $\Omega_C^k(N)$ is annihilated by Z which implies $J_B\Omega_C^k(N) = J_C\Omega_C^k(N)$.The module $B_{n0} \cong C_{n0} \cong K$.

There are isomorphisms:

$Hom_{B_n}(\Omega_C^k(N), B_{n0}) \cong Hom_{B_{n0}}(\Omega_C^k(N)/J_B\Omega_C^k(N), B_{n0}) \cong$
$Hom_{C_{n0}}(\Omega_C^k(N)/J_c\Omega_C^k(N), C_{n0}) \cong Hom_{C_n}(\Omega_C^k(N), C_{n0}) \cong Ext_{C_n}^k(N, C_{n0}).$

We then have an exact sequence: *) $0 \to F_{C_n}(N) \xrightarrow{\alpha} F_{B_n}(N) \to \overset{m}{\underset{k=1}{\oplus}} S^{n_{k-1}}[k] \to 0$

Lemma 8. *The map α is a morphism of $C_n^!$-modules.*

Proof. Let x be an element of $Ext_{C_n}^k(K,K)$ and $y \in Ext_{C_n}^k(N,K)$ we want to prove $\alpha(xy) = x\alpha(y)$.

The element x is an extension: $0 \to K \to E \to K \to 0$ and $y : 0 \to K \to V \to \Omega_C^{k-1}(N) \to 0$, the induced map f given below corresponds to y :

$$\begin{array}{ccccccc} 0 \to & \Omega_C^k(N) & \to & C_n^{n_{k-1}} & \to & \Omega_C^{k-1}(N) & \to 0 \\ & \downarrow f & & \downarrow & & \downarrow 1 & \\ 0 \to & K & \to & V & \to & \Omega_C^{k-1}(N) & \to 0 \end{array}$$

Consider the following pull back:

$$\begin{array}{ccccccc} 0 \to & K & \to & L & \to & \Omega_C^k(N) & \to 0 \\ & \downarrow 1 & & \downarrow & & \downarrow f & \\ 0 \to & K & \to & E & \to & K & \end{array}$$

The exact sequence: $0 \to B_n^{n_{k-1}}[-k] \to \Omega_B^k(N) \xrightarrow{\pi_k} \Omega_C^k(N) \to 0$ induces a pull back of B_n-modules:

$$\begin{array}{ccccccc} 0 \to & K & \to & W & \to & \Omega_B^k(N) & \to 0 \\ & \downarrow 1 & & \downarrow & & \downarrow \pi_k & \\ 0 \to & K & \to & L & \to & \Omega_C^k(N) & \end{array}$$

It was proved above the existence of commutative exact diagrams:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & 0 & \to & B_n^{n_{k-2}} & \to & B_n^{n_{k-2}} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Omega_B^k(N) & \to & B_n^{n_{k-2}+n_{k-1}} & \to & \Omega_B^{k-1}(N) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Omega_B^k(N) & \to & B_n^{n_{k-1}} & \to & \Omega_C^{k-1}(N) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

and

$$\begin{array}{ccccccc} 0 \to & \Omega_B^k(N) & \to & B_n^{n_{k-1}} & \to & \Omega_C^{k-1}(N) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow 1 & \\ 0 \to & \Omega_C^k(N) & \to & C_n^{n_{k-1}} & \to & \Omega_C^{k-1}(N) & \to 0 \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array}$$

Gluing diagrams we obtain a commutative diagram with exact rows:

$$\begin{array}{cccccccccccc}
 & 0\to & \Omega_B^{k+1}(\mathrm{N}) & \to & \mathrm{B}_n^{n_k+n_{k-1}} & \to & \mathrm{B}_n^{n_{k-2}+n_{k-1}} & \to & \Omega_B^{k-1}(\mathrm{N}) & \to 0 \\
 & & \varphi\downarrow & & \downarrow & \searrow & \downarrow & & \downarrow & \\
\alpha(\mathrm{xy}): & 0\to & \mathrm{K} & \to & \mathrm{W} & \to & \mathrm{B}_n^{n_{k-1}} & \to & \Omega_C^{k-1}(\mathrm{N}) & \to 0 \\
 & & \downarrow 1 & & \downarrow & & \downarrow & & \downarrow 1 & \\
\mathrm{xy}: & 0\to & \mathrm{K} & \to & \mathrm{L} & \to & \mathrm{C}_n^{n_{k-1}} & \to & \Omega_C^{k-1}(\mathrm{N}) & \to 0 \\
 & & \downarrow 1 & & \downarrow & & \downarrow & & \downarrow 1 & \\
\mathrm{xy}: & 0\to & \mathrm{K} & \to & \mathrm{E} & \to & \mathrm{V} & \to & \Omega_C^{k-1}(\mathrm{N}) & \to 0
\end{array}$$

$\alpha(xy) = \phi \in Ext_{B_n}^{k+1}(N, K)$.

In the other hand we have the following commutative diagram with exact rows:

$$\begin{array}{cccccccc}
0\to & \Omega_B^{k+1}(\mathrm{N}) & \to & \mathrm{B}_n^{n_k+n_{k-1}} & \to & \Omega_B^{k}(\mathrm{N}) & \to 0 \\
 & \downarrow 1 & & \downarrow & & \downarrow \pi_k & \\
0\to & \Omega_B^{k+1}(\mathrm{N}) & \to & \mathrm{B}_n^{n_k} & \to & \Omega_C^{k}(\mathrm{N}) & \to 0 \\
 & \downarrow \pi_{k+1} & & \downarrow & & \downarrow 1 & \\
0\to & \Omega_C^{k+1}(\mathrm{N}) & \to & \mathrm{C}_n^{n_k} & \to & \Omega_C^{k}(\mathrm{N}) & \to 0 \\
 & \downarrow \varphi' & & \downarrow & & \downarrow 1 & \\
0\to & \mathrm{K} & \to & \mathrm{L} & \to & \Omega_C^{k}(\mathrm{N}) & \to 0 \\
 & \downarrow 1 & & \downarrow & & \downarrow \mathrm{f} & \\
0\to & \mathrm{K} & \to & \mathrm{E} & \to & \mathrm{K} & \to 0
\end{array}$$

Gluing diagrams we obtain the following commutative exact diagrams:

$$\begin{array}{cccccccc}
0\to & \Omega_B^{k+1}(\mathrm{N}) & \to & \mathrm{B}_n^{n_k+n_{k-1}} & \to & \Omega_B^{k}(\mathrm{N}) & \to 0 \\
 & \downarrow \varphi'\pi_{k+1} & & \downarrow & & \downarrow \mathrm{f}\pi_k & \\
0\to & \mathrm{K} & \to & \mathrm{E} & \to & \mathrm{K} & \to 0
\end{array}$$

and

$$\begin{array}{cccccccc}
0\to & \Omega_B^{k+1}(\mathrm{N}) & \to & \mathrm{B}_n^{n_k+n_{k-1}} & \to & \Omega_B^{k}(\mathrm{N}) & \to 0 \\
 & \downarrow \varphi & & \downarrow & & \downarrow \mathrm{f}\pi_k & \\
0\to & \mathrm{K} & \to & \mathrm{E} & \to & \mathrm{K} & \to 0
\end{array}$$

The map φ' corresponds with $xf \in Ext_{C_n}^{k+1}(N, K)$, $\alpha(y) = \pi_k f$, $x\alpha(y) = \varphi$.

Then we have: $\alpha(xy) = \alpha(xf) = \varphi'\pi_{k+1} = \Omega(f\pi_k) = \varphi = x\alpha(y)$. □

Lemma 9. *There is an isomorphism:* $B_n^! F_C(M) = F_B(M)$.

Proof. Let x be an element of $Ext_{B_n}^k(M, K)$ and φ the corresponding morphism: $\varphi : \Omega_B^k(M) \to K$.As above there exists the following commutative exact diagram:

$$\begin{array}{cccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & 0 & \to & B_n^{n_{k-2}}[-k+1] & \to & B_n^{n_{k-2}}[-k+1] & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & \Omega_B^k(M) & \to & B_n^{n_{k-2}+n_{k-1}}[-k+1] & \to & \Omega_B^{k-1}(M) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & \Omega_B\Omega_C^{k-1}(M) & \to & B_n^{n_{k-1}}[-k+1] & \to & \Omega_C^{k-1}(M) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

Since the module $\Omega_B^k(M)$ is generated in degree k,there exists an exact sequence of graded modules generated in degree k:

$0 \to \Omega_B^k(M) \to JB_n^{n_{k-1}}[-k+1] \to J\Omega_C^{k-1}(M) \to 0$, which in turn induces an exact sequence: $0 \to J\Omega_B^k(M) \to J^2B_n^{n_{k-1}}[-k+1] \to J^2\Omega_C^{k-1}(M) \to 0$ and there

exists a commutative exact diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0\rightarrow & & \mathrm{J}\Omega^k_B(\mathrm{M}) & \rightarrow & \mathrm{J}^2\mathrm{B}_n^{n_{k-1}} & \rightarrow & \mathrm{J}^2\Omega^{k-1}_C(\mathrm{M}) & & \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0\rightarrow & & \Omega^k_B(\mathrm{M}) & \stackrel{j}{\rightarrow} & \mathrm{JB}_n^{n_{k-1}} & \rightarrow & \mathrm{J}\Omega^{k-1}_C(\mathrm{M}) & & \rightarrow 0 \\ & & \downarrow\pi & & \downarrow\overline{\pi} & & \downarrow & & \\ 0\rightarrow & & \Omega^k_B(\mathrm{M})/\mathrm{J}\Omega^k_B(\mathrm{M}) & \underset{\overline{q}}{\overset{\overline{j}}{\rightleftarrows}} & \mathrm{JB}_n^{n_{k-1}}/\mathrm{J}^2\mathrm{B}_n^{n_{k-1}} & \rightarrow & \mathrm{J}\Omega^{k-1}_C(\mathrm{M})/\mathrm{J}^2\Omega^{k-1}_C(\mathrm{M}) & & \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Since $\overline{q}\overline{j}=1$, it follows $\overline{q\pi}j=\overline{q}\overline{j}\pi=\pi$.

Being K semisimple, the map φ factors as follows:

$$\begin{array}{ccccc} \Omega^k_B(M) & & \stackrel{\varphi}{\rightarrow} & & K \\ & \pi\searrow & & \nearrow t & \\ & & \Omega^k_B(M)/J\Omega^k_B(M) & & \end{array}.$$

Set $f=t\overline{q\pi}$, $f:JB_n^{n_{k-1}}[-k+1]\rightarrow K$. Then $fj=t\overline{q\pi}j=t\pi=\varphi$.

Consider the commutative diagram with exact rows:

$$\begin{array}{cccccccc} & 0\rightarrow & \Omega_B\Omega^{k-1}_C(M) & \rightarrow & B_n^{n_{k-1}} & \rightarrow & \Omega^{k-1}_C(M) & \rightarrow 0 \\ & & \downarrow j & & \downarrow 1 & & \downarrow\rho & \\ & 0\rightarrow & JB^{n_{k-1}}[-k+1] & \rightarrow & B_n^{n_{k-1}} & \rightarrow & \Omega^{k-1}_C(M)/J\Omega^{k-1}_C(M) & \rightarrow 0 \\ & & \downarrow f & & \downarrow & & \downarrow\cong & \\ x: & 0\rightarrow & K & \rightarrow & E & \rightarrow & \bigoplus_{n_{k-1}}K & \rightarrow 0 \end{array}.$$

The map corresponding to the last column is: $p=\begin{bmatrix} p_1 \\ \vdots \\ p_{n_{k-1}}\end{bmatrix}:\Omega^{k-1}_C(M)\rightarrow\bigoplus_{n_{k-1}}K$.

Each $p_i:\Omega^{k-1}_C(M)\rightarrow K$ corresponds to an element of $Ext^{k-1}_{C_n}(M,K)$.

$x=(x_1,x_2...x_{n_{k-1}})$ and each x_i is an extension: $0\rightarrow K\rightarrow E_i\rightarrow K\rightarrow 0$.Taking pull backs:

$$\begin{array}{ccccccccc} x_ip_i: & 0\rightarrow & K & \rightarrow & L_i & \rightarrow & \Omega^{k-1}_C(M) & \rightarrow 0 \\ & & \downarrow 1 & & \downarrow & & \downarrow p_i & \\ x_i: & 0\rightarrow & K & \rightarrow & E_i & \rightarrow & K & \rightarrow 0 \end{array}$$

where each $x_ip_i\in B_n^!Ext^{k-1}_{C_n}(M,K)$ and $xp=\sum x_ip_i\in B_n^!Ext^{k-1}_{C_n}(M,K)$.

There is also the following induced diagram with exact rows:

$$\begin{array}{cccccccc} & 0\rightarrow & \Omega_B\Omega^{k-1}_C(M) & \rightarrow & B_n^{n_{k-1}}[k+1] & \rightarrow & \Omega^{k-1}_C(M) & \rightarrow 0 \\ & & \downarrow h & & \downarrow & & \downarrow 1 & \\ & 0\rightarrow & K & \rightarrow & L & \rightarrow & \Omega^{k-1}_C(M) & \rightarrow 0 \\ & & \downarrow 1 & & \downarrow & & \downarrow p & \\ x: & 0\rightarrow & K & \rightarrow & E & \rightarrow & \bigoplus_{n_{k-1}}K & \rightarrow 0 \end{array}$$

Gluing the diagrams we obtain the following commutative exact diagram:

$$\begin{array}{cccccccc} & 0\rightarrow & \Omega_B\Omega^{k-1}_C(M) & \rightarrow & B_n^{n_{k-1}}[k+1] & \rightarrow & \Omega^{k-1}_C(M) & \rightarrow 0 \\ & & \downarrow h & & \downarrow & & \downarrow p & \\ x: & 0\rightarrow & K & \rightarrow & E & \rightarrow & \bigoplus_{n_{k-1}}K & \rightarrow 0 \end{array}$$

It follows $h=fj=\varphi$ up to homotopy.

We have proved $B_n^!Ext^{k-1}_{C_n}(M,K)=Ext^k_{B_n}(M,K)$.

It follows by induction $B_n^!F_C(M)=F_B(M)$. □

We can prove now the following:

Proposition 10. *Let M be a Koszul non semisimple B_n-module with $ZM = 0$, $F_B : K_B \to K_{B^!}$, $F_C : K_C \to K_{C^!}$ Koszul dualities. Then there is an isomorphism of $B_n^!$-modules: $B_n^! \underset{C^!}{\otimes} F_C(M) \cong F_B(M)$.*

Proof. We proved in the previous lemma that the map $\mu : B_n^! \underset{C^!}{\otimes} F_C(M) \to F_B(M)$ given by multiplication is surjective and we know that $B_n^! = C_n^! \oplus ZC_n^!$, so there is a splittable sequence of $C_n^!$-modules: $0 \to C_n^! \to B_n^! \to ZC_n^! \to 0$ which induces a commutative exact diagram:

$$\begin{array}{ccccccccc} 0 \to & C_n^! \underset{C^!}{\otimes} F_C(M) & \to & B_n^! \underset{C^!}{\otimes} F_C(M) & \to & ZC_n^! \underset{C^!}{\otimes} F_C(M) & \to 0 \\ & \downarrow\cong & & \downarrow \mu & & \downarrow \mu'' & \\ 0 \to & F_C(M) & \to & F_B(M) & \to & \overset{m}{\oplus} S^{n_{k-1}}[k] & \to 0 \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

By dimensions μ'' is an isomorphism, therefore μ is an isomorphism. $\square$

It as proved in $[MM]$, $[MS]$ that the duality $\phi : gr_{B_n^!} \to \mathcal{LCP}_{B_n}$, with $\mathcal{LCP}_{B_n}$ the category of linear complexes of graded projective B_n-modules, induces a duality of triangulated categories $\overline{\phi} : \underline{gr}_{B_n^!} \to D^b(Qgr_{B_n^{op}})$. In particular given a complex πX° in $D^b(Qgr_{B_n^{op}})$, there is a totally linear complex (see $[MM]$ for definition) Y° such that πY° is isomorphic to πX°, Moreover, Y° is quasi isomorphic to a linear complex of projectives P° and by $[MS]$ $P^\circ = \phi(M)$.Therefore πX° is quasi isomorphic to $\pi\phi(M)$.

It was proved in $[MS]$ that $H^i(\phi(M)) = 0$ for all $i \neq 0$ if and only if M is Koszul and in this case if $G_{B_n^!} : K_{B_n^!} \to K_{B_n}$ is Koszul duality, then $H^0(\phi(M)) \cong G_{B_n^!}(M)$.

Since $B_n^!$ is a finite dimensional algebra, it follows by $[MZ]$ that for any finitely generated $B_n^!$-module M there exists an integer $k \geq 0$ such that $\Omega^k M$ is weakly Koszul.

since Ω is the shift in the triangulated category $\underline{gr}_{B_n^!}$ and $\overline{\phi}$ is a duality it follows $\overline{\phi}(\Omega^k M) \cong \overline{\phi}(M)[k]$. Being the category $\mathcal{T}$ triangulated, it is invariant under shift and $\pi\phi(M) \in \mathcal{T}$ if and only if $\pi\phi(\Omega^k M) \in \mathcal{T}$.

We may assume M is weakly Koszul and $M = \sum\limits_{i\geq 0} M_i$, $M_0 \neq 0$. By $[MZ]$ there exists an exact sequence: $0 \to K_M \to M \to L \to 0$ with $K_M < M_0 >$ generated by the degree zero part of M, K_M is Koszul and $J^j K_M = J^j M \cap K_M$ for all $j > 0$.

Being ϕ an exact functor there is an exact sequence: $0 \to \phi(L) \to \phi(M) \to \phi(K_M) \to 0$ of complexes of B_n-modules, which induces a long exact sequence:

$... \to H^1(\phi(L)) \to H^1(\phi(M)) \to H^1(\phi(K_M)) \to H^0(\phi(L)) \to H^0(\phi(M)) \to H^0(\phi(K_M)) \to 0$

where $H^0(\phi(M)) \cong H^0(\phi(K_M))$ and $H^i(\phi(L)) \cong H^i(\phi(M)$ for all $i \neq 0$. Being K_M Koszul $H^0(\phi(K_M)) \cong G_{B_n^!}(K_M)$ and $G_{B_n^!}(K_M)$ is of Z-torsion.

According to $[MZ]$ there is a filtration: $M = U_p \supset U_{p-1} \supset ...U_1 \supset U_0 = K_M$ such that $U_i/U_{i-1}$is Koszul and $J^k U_i \cap U_{i-1} = J^k U_{i-1}$.

The module L is weakly Koszul and it has a filtration: $L = U_p/U_0 \supset U_{p-1}/U_0 \supset ...U_1/U_0$ with factors Koszul, it follows by induction each $G_{B_n^!}(U_i/U_{i-1}) = V_i$ is a Koszul B_n-module of Z-torsion.

Each V_i has a filtration: $V_i \supset ZV_i \supset Z^2V_i \supset ... \supset Z^{k_i}V_i \supset 0$, $Z^{k_i}V_i \neq 0$, $Z^{k_i+1}V_i = 0$. After a truncation $V_{i\geq n_i}$ we may assume all Z^jV_i Koszul. But $V_{i\geq n_i} = J^{n_i}V_i \cong G_{B_n^!}(\Omega^{n_i}(U_i/U_{i-1}))$. Taking $n = \max\{n_i\}$ we change M for $\Omega^n(M)$,which is weakly Koszul with filtration: $\Omega^n M = \Omega^n U_p \supset \Omega^n U_{p-1} \supset ...\Omega^n U_1 \supset \Omega^n U_0$.

We may assume all Z^jV_i are Koszul. There exist exact sequences:

*) $0 \to F_{B_n}(Z^{k_i}V_i)[-k_i] \to F_{B_n}(Z^{k_i-1}V_i/Z^{k_i}V_i)[-k_i+1]... \to$
$F_{B_n}(V_i/ZV_i) \to U_i/U_{i-1} \to 0$

where each $F_{B_n}(Z^jV_i/Z^{j+1}V_i) \cong B_n^! \underset{C_n^!}{\otimes} X_{ij}$ is an induced module of a Koszul $C_n^!$-module X_{ij}.

Lemma 10. *Let R be a Z-graded K-algebra, with K a field, M a graded left R-module and N a graded right R-module. Then $M \underset{R}{\otimes} N$ is a graded K-module such that $M \underset{R}{\otimes} N[j] \cong (M \underset{R}{\otimes} N)[j]$ as graded K-modules.*

Proof. Recall the definition of the graded tensor product $[Mac]$:

Let $\psi : M \underset{K}{\otimes} R \underset{K}{\otimes} N \to M \underset{K}{\otimes} N$ be the map: $\psi(m \otimes r \otimes n) = mr \otimes n - m \otimes rn$. Then $Cok\psi = M \underset{R}{\otimes} N$.

The K- module $M \underset{K}{\otimes} N$ has grading: $(M \underset{K}{\otimes} N)_k = \sum\limits_{i+j=k} M_i \underset{K}{\otimes} N_j$. It follows $M \underset{K}{\otimes} N\ [j] \cong (M \underset{K}{\otimes} N\ )[j]$.

and there is an isomorphism of exact sequences:

$$\begin{array}{ccccccc}
M \underset{K}{\otimes} R \underset{K}{\otimes} (N[j]) & \to & M \underset{K}{\otimes} (N[j]) & \to & M \underset{R}{\otimes} (N[j]) & \to 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
(M \underset{K}{\otimes} R \underset{K}{\otimes} N)[j] & \to & (M \underset{K}{\otimes} N)[j] & \to & (M \underset{R}{\otimes} N)[j] & \to 0
\end{array}$$

□

Lemma 11. *Let $B_n^!$ and $C_n^!$ be the algebras given above, for any finitely generated graded $C_n^!$-module M there is an isomorphism: $\Omega_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M) \cong B_n^! \underset{C_n^!}{\otimes} \Omega_{C_n^!}(M)$.*

Proof. Let $0 \to \Omega_{C_n^!}(M) \to F \to M \to 0$ be an exact sequence with F free of rank r, the graded projective cover of M. Then $\Omega_{C_n^!}(M) \subset J_{C_n^!}F$.

We proved $B_n^! = C_n^! \oplus ZC_n^!$, therefore $J_{B_n^!} = J_{C_n^!} + ZC_n^!$. It follows: $B_n^! \underset{C_n^!}{\otimes} J_{C_n^!} = C_n^! \underset{C_n^!}{\otimes} J_{C_n^!} + ZC_n^! \underset{C_n^!}{\otimes} J_{C_n^!} = J_{C_n^!} + Z \underset{C_n^!}{\otimes} J_{C_n^!} \subset J_{B_n^!}$.

Therefore: $B_n^! \underset{C_n^!}{\otimes} \Omega_{C_n^!}(M) \subset B_n^! \underset{C_n^!}{\otimes} \underset{r}{\oplus} J_{C_n^!} \cong \underset{r}{\oplus} B_n^! \underset{C_n^!}{\otimes} J_{C_n^!} \subset \underset{r}{\oplus} J_{B_n^!} \cong J_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} F)$.

It follows: $0 \to B_n^! \underset{C_n^!}{\otimes} \Omega_{C_n^!}(M) \to B_n^! \underset{C_n^!}{\otimes} F \to B_n^! \underset{C_n^!}{\otimes} M \to 0$ is exact and $B_n^! \underset{C_n^!}{\otimes} F$ is the graded projective cover of $B_n^! \underset{C_n^!}{\otimes} M$. Then $\Omega_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M) \cong B_n^! \underset{C_n^!}{\otimes} \Omega_{C_n^!}(M)$. □

Lemma 12. *Let $B_n^!$ and $C_n^!$ be the algebras given above and let M be a Koszul $C_n^!$-module. Then $B_n^! \underset{C_n^!}{\otimes} M$ is Koszul and $G_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M) \cong G_{C_n^!}(M)$..*

Proof. Let $...\to F_n[-n] \to F_{n-1}[-n+1] \to ...F_1[-1] \to F_0 \to M \to 0$ be a graded projective resolution of M,with each F_i free of rank r_i.Tensoring with $B_n^! \underset{C_n^!}{\otimes}$ we obtain a graded projective resolution of $B_n^! \underset{C_n^!}{\otimes} M : \to (B_n^! \underset{C_n^!}{\otimes} F_n)[-n] \to (B_n^! \underset{C_n^!}{\otimes}$

$F_{n-1})[-n+1] \to ...(B_n^! \underset{C_n^!}{\otimes} F_1)[-1] \to B_n^! \underset{C_n^!}{\otimes} F_0 \to B_n^! \underset{C_n^!}{\otimes} M \to 0$ with each $B_n^! \underset{C_n^!}{\otimes} F_i$ free $B_n^!$-modules of rank r_i.

Moreover, $Ext^n_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M, K) \cong Hom_{B_n^!}(\Omega^n\ (B_n^! \underset{C_n^!}{\otimes} M), K) \cong Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} \Omega^n M, K) \cong Hom_{C_n^!}(\Omega^n M, K) \cong Ext^n_{C_n^!}(M, K)$.

Therefore: $G_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M) \cong G_{C_n^!}(M)$. □

Remark 1. *To be $G_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M)$ a C_n-module means: $ZG_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M) = 0$.*

We know $B_n \cong \underset{m\geq 0}{\oplus} Ext^m_{B_n^!}(K, K)$, $C_n \underset{m\geq 0}{\oplus} Ext^m_{C_n^!}(K, K)$, and $B_n/ZB_n \cong C_n$. Since $C_n^!$ is a sub algebra of $B_n^!$, given an extension $x : 0 \to K \to E_1 \to E_2 \to ...E_n \to K \to 0$ of $B_n^!$, we obtain by restriction of scalars an extension $resx$: $0 \to resK \to resE_1 \to resE_2 \to ...resE_n \to resK \to 0$ of $C_n^!$-modules, where $resM$ is the module M with multiplication of scalars restricted to $C_n^!$.It is clear $res(xy) = res(x)res(y)$ and restriction gives an homomorphism of graded k-algebras: res : $\underset{m\geq 0}{\oplus} Ext^m_{B_n^!}(K, K) \to \underset{m\geq 0}{\oplus} Ext^m_{C_n^!}(K, K)$.

Lemma 13. *There is an homomorphism: $\rho : Ext^1_{B_n^!}(K, K) \to Ext^1_{B_n^!}((B_n^! \underset{C_n^!}{\otimes} K, K)$, given by the Yoneda product $\rho(x) = x\mu$ (pull back) of the exact sequence x with the multiplication map $\mu : B_n^! \underset{C_n^!}{\otimes} K \to K$, such that the composition of the map, $\psi_1 : Ext^1_{B_n^!}((B_n^! \underset{C_n^!}{\otimes} K, K) \to Ext^1_{C_n^!}(K, K)$ in the previous lemma with ρ, is the restriction: $\psi\rho$=res.*

Proof. Let x be the extension: $x : 0 \to K \to E \to K \to 0$. Since $B_n^!$ is a free $C_n^!$-module, there is a commutative exact diagram

$$\begin{array}{ccccccc}
0 \to & B_n^! \underset{C_n^!}{\otimes} K & \to & B_n^! \underset{C_n^!}{\otimes} E & \to & B_n^! \underset{C_n^!}{\otimes} K & \to 0 \\
& \downarrow \mu & & \downarrow \mu & & \downarrow \mu & \\
0 \to & K & \to & E & \to & K & \to 0
\end{array}$$

with μ multiplication.

This diagram splits in two diagrams:

$$\begin{array}{ccccccc}
0 \to & B_n^! \underset{C_n^!}{\otimes} K & \to & B_n^! \underset{C_n^!}{\otimes} E & \to & B_n^! \underset{C_n^!}{\otimes} K & \to 0 \\
& \downarrow \mu & & \downarrow & & \downarrow 1 & \\
0 \to & K & \to & W & \to & B_n^! \underset{C_n^!}{\otimes} K & \to 0 \\
& \downarrow 1 & & \downarrow & & \downarrow \mu & \\
0 \to & K & \to & E & \to & K & \to 0
\end{array}$$

Then $\rho(x) = x\mu = \mu(B_n^! \otimes x)$.

For any finitely generated $C_n^!$-module M there is an isomorphism α obtained as the composition of the natural isomorphisms:

$Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M, K) \cong Hom_{C_n^!}(M, Hom_{B_n^!}(B_n^!, K)) \cong Hom_{C_n^!}(M, K)$.

If $j : M \to B_n^! \underset{C_n^!}{\otimes} M$ be the map $j(m) = 1 \otimes m$ and $f : B_n^! \underset{C_n^!}{\otimes} M \to K$ is any map, then $\alpha(f) = fj$.

Then $\psi\rho(x) = \psi(x\mu)$ is the top sequence in the commutative exact diagram:

$$\begin{array}{ccccccccc}
0 \to & K & \to & L & \to & K & \to 0 \\
& \downarrow 1 & & \downarrow & & \downarrow j & \\
0 \to & K & \to & W & \to & B_n^! \underset{C_n^!}{\otimes} K & \to 0 \\
& \downarrow 1 & & \downarrow & & \downarrow \mu & \\
0 \to & K & \to & E & \to & K & \to 0
\end{array}$$

Since $\mu j = 1$,gluing both diagrams we obtain $\psi\rho(x) = resx$. □

Lemma 14. *Under the conditions of the previous lemma the map ρ is surjective.*

Proof. Since $B_n^! = C_n^! \oplus ZC_n^!$, $B_n^! \underset{C_n^!}{\otimes} K$ is a graded vector space of dimension two with one copy of K in degree zero and one copy of K in degree one. Hence the multiplication map $\mu : B_n^! \underset{C_n^!}{\otimes} K \to K$ is an epimorphsi with kernel $u : K[-1] \to B_n^! \underset{C_n^!}{\otimes} K$.

Let $y : 0 \to K[-1] \to E \to B_n^! \underset{C_n^!}{\otimes} K \to 0$ be an element of $Ext^1_{B_n^!}(K, B_n^! \underset{C_n^!}{\otimes} K)$ and take the pullback:

$$\begin{array}{ccccccc}
0 \to & K[-1] & \to & N & \to & K[-1] & \to 0 \\
& \downarrow 1 & & \downarrow & & \downarrow u & \\
0 \to & K[-1] & \to & E & \to & B_n^! \underset{C_n^!}{\otimes} K & \to 0
\end{array}$$

But the top exact sequence split because the ends are generated in the same degree and the algebra is Koszul or equivalently there is a lifting $v : K[-1] \to E$ of u and we get a commutaive exact diagram:

$$\begin{array}{ccccccc}
& 0 & \to & K[-1] & \to & K[-1] & \to 0 \\
& \downarrow 1 & & \downarrow v & & \downarrow u & \\
0 \to & K[-1] & \to & E & \to & B_n^! \underset{C_n^!}{\otimes} K & \to 0 \\
& \downarrow 1 & & \downarrow & & \downarrow \mu & \\
0 \to & K[-1] & \to & L & \to & K & \to 0
\end{array}$$

Proving ρ is surjective. □

Corollary 8. *The map $res : \underset{m\geq 0}{\oplus} Ext^m_{B_n^!}(K, K) \to \underset{m\geq 0}{\oplus} Ext^m_{C_n^!}(K, K)$ is a surjective homomorphism of algebras and the kernel of res is the ideal $ZB_n = B_nZ$.*

Proof. Since both $B_n^!$ and $C_n^!$ are Koszul algebras they are graded algebras generated in degree one, and it follows from the lemma that for any $m > 0$ the map $res : Ext^m_{B_n^!}(K, K) \to Ext^m_{C_n^!}(K, K)$ is surjective.

Observe that for any homomorphism $f : B_n \to C_n$ Z is in the kernel.

We have in B_n the equality $X_1\delta_1 - \delta_1 X_1 = Z^2$. Since C_n is commutative, $f(X_1\delta_1 - \delta_1 X_1) = f(X_1)f(\delta_1) - f(\delta_1)f(X_1) = f(Z)^2 = 0$,

But since C_n is an integral domain, it follows $f(Z) = 0$.

In particular $ZB_n \subseteq Ker(res)$ and there is a factorization:
$$\begin{array}{ccc}
B_n & \to & C_n \\
\searrow & & \nearrow \alpha \\
& B_n/ZB_n &
\end{array}$$
and since $B_n/ZB_n \cong C_n$ it follows by dimension, that α is an isomorphism. □

Lemma 15. *With the same notation as in the previous lemma, let M be a Koszul $C_n^!$-module and $\psi : G_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M) \to G_{C_n^!}(M)$, the isomorphism in the previous lemma.*

Then given $y \in Ext^m_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M, K)$ and $c \in Ext^1_{B_n^!}(K, K)$, we have $\psi(cy) = res(c)\psi(y)$.

Proof. The map $f : B_n^! \underset{C_n^!}{\otimes} \Omega^m(M) \to K$ corresponding to the extension y is the map in the commutative exact diagram:

$$\begin{array}{ccccccccccc} 0 \to & B_n^! \underset{C_n^!}{\otimes} \Omega^m(\mathrm{M}) & \to & B_n^! \underset{C_n^!}{\otimes} \mathrm{C}_n^{!k_{m-1}} & \ldots \to & B_n^! \underset{C_n^!}{\otimes} \mathrm{C}_n^{!k_0} & \to & B_n^! \underset{C_n^!}{\otimes} \mathrm{M} & \to 0 \\ & \downarrow f & & \downarrow & & \downarrow & & \downarrow 1 & \\ 0 \to & K & \to & E_1 & \ldots \to & E_m & \to & B_n^! \underset{C_n^!}{\otimes} \mathrm{M} & \to 0 \end{array}$$

where y is the bottom raw.

If $j : \Omega^m(M) \to B_n^! \underset{C_n^!}{\otimes} \Omega^m(M)$ is the map $j(m) = 1 \otimes m$, then $\psi(y)$ is the extension corresponding to the map fj.

Consider the commutative diagram with exact raws:

$$\begin{array}{ccccccc} 0 \to & \Omega^{m+1}(M) & \to & C_n^{!k_m} & \to & \Omega^m(M) & \to 0 \\ & j \downarrow & & j \downarrow & & j \downarrow & \\ 0 \to & B_n^! \underset{C_n^!}{\otimes} B\Omega^{m+1}(M) & \to & B_n^! \underset{C_n^!}{\otimes} C_n^{!k_m} & \to & B_n^! \underset{C_n^!}{\otimes} \Omega^m(M) & \to 0 \\ & \Omega f \downarrow & & \downarrow & & f \downarrow & \\ 0 \to & JB_n^! & \to & B_n^! & \to & K & \to 0 \\ & g \downarrow & & \downarrow & & 1 \downarrow & \\ 0 \to & K & \to & L & \to & K & \to 0 \end{array}$$

where c is the bottom sequence.

Since $B_n^! = C_n^! \oplus C_n^! Z$ as $C_n^!$-module, the map g restricted to $C_n^!$ represents the extension $res(c)$.

Taking the pullback we obtain a commutative diagram with exact raws:

$$\begin{array}{ccccccc} 0 \to & \Omega^{m+1}(M) & \to & C_n^{!k_m} & \to & \Omega^m(M) & \to 0 \\ & g\Omega(f)j \downarrow & & \downarrow & & 1 \downarrow & \\ 0 \to & K & \to & W & \to & \Omega^m(M) & \to 0 \\ & 1 \downarrow & & \downarrow & & fj \downarrow & \\ 0 \to & K & \to & L & \to & K & \to 0 \end{array}$$

and $\Omega(fj) = \Omega(f)j$.

It follows $\psi(cy) = res(c)\psi(y)$. □

As a corollary we obtain the following:

Proposition 11. *Let M be a Koszul $C_n^!$-module and $G_{B_n^!} = \underset{m \geq 0}{\oplus} Ext^m_{C_n^!}(-, K)$ Koszul duality. Then $Z(G_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M)) = 0$.*

Proof. Denote by z the extension corresponding to Z under the isomorphism $B_n \cong \underset{m \geq 0}{\oplus} Ext^m_{B_n^!}(K, K)$. By the previous lemma, for any extension $y \in Ext^m_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M), K)$, $\psi(zy) = res(z)\psi(y)$ and by lemma ?, $res(z) = 0$. Since ψ is an isomorphism, it follows $zy = 0$, hence $Z(G_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M)) = 0$. □

Proposition 12. *Let $B_n^!$ and $C_n^!$ be the algebras given above. Then for any induced module $B_n^! \underset{C_n^!}{\otimes} M$ when we apply the duality $\overline{\phi}$ to $B_n^! \underset{C_n^!}{\otimes} M$ we obtain an element of $\mathcal{T}$.*

Proof. There exists some integer $n \geq 0$ such that $\Omega^n M$ and $\Omega^n(B_n^! \underset{C_n^!}{\otimes} M)$ are weakly Koszul. Since $\overline{\phi}(\Omega^n(B_n^! \underset{C_n^!}{\otimes} M)) \cong \overline{\phi}(B_n^! \underset{C_n^!}{\otimes} M)[n]$.The object $\overline{\phi}(\Omega^n(B_n^! \underset{C_n^!}{\otimes} M))$ is in $\mathcal{T}$ if and only if $\overline{\phi}(B_n^! \underset{C_n^!}{\otimes} M)$ is in $\mathcal{T}$. We may assume M and $B_n^! \underset{C_n^!}{\otimes} M$ are weakly Koszul.

The module M has a filtration: $M = U_p \supset U_{p-1} \supset ...U_1 \supset U_0$ such that U_i/U_{i-1} is Koszul, hence; $B_n^! \underset{C_n^!}{\otimes} M$ has a filtration: $B_n^! \underset{C_n^!}{\otimes} M = B_n^! \underset{C_n^!}{\otimes} U_p \supset B_n^! \underset{C_n^!}{\otimes} U_{p-1} \supset ...B_n^! \underset{C_n^!}{\otimes} U_1 \supset B_n^! \underset{C_n^!}{\otimes} U_0$ such that $B_n^! \underset{C_n^!}{\otimes} U_i/B_n^! \underset{C_n^!}{\otimes} U_{i-1} \cong B_n^! \underset{C_n^!}{\otimes} U_i/U_{i-1}$ is Koszul.

The exact sequence: $0 \to B_n^! \underset{C_n^!}{\otimes} U_0 \to B_n^! \underset{C_n^!}{\otimes} U_1 \to B_n^! \underset{C_n^!}{\otimes} U_1/U_0 \to 0$ induces an exact sequence of complexes: $0 \to \phi(\ B_n^! \underset{C_n^!}{\otimes} U_1/U_0) \to \phi(B_n^! \underset{C_n^!}{\otimes} U_1) \to \phi(B_n^! \underset{C_n^!}{\otimes} U_0) \to 0$which induces a long exact sequence:

$...\to H^1(\phi(\ B_n^! \underset{C_n^!}{\otimes} U_1/U_0)) \to H^1(\phi(B_n^! \underset{C_n^!}{\otimes} U_1)) \to H^1(\phi(B_n^! \underset{C_n^!}{\otimes} U_0)) \to H^0(\phi(\ B_n^! \underset{C_n^!}{\otimes} U_1/U_0)) \to H^0(\phi(B_n^! \underset{C_n^!}{\otimes} U_1)) \to H^0(\phi(B_n^! \underset{C_n^!}{\otimes} U_0)) \to 0$

where $H^i(\phi(B_n^! \underset{C_n^!}{\otimes} U_0)) = 0$ for $i \neq 0$ and $H^0(\phi(B_n^! \underset{C_n^!}{\otimes} U_0)) = G_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} U_0) \cong G_{C_n^!}(U_0)$ of Z-torsion, $H^0(\phi(B_n^! \underset{C_n^!}{\otimes} U_1)) \cong H^0(\phi(B_n^! \underset{C_n^!}{\otimes} U_0))$ and $H^i(\phi(\ B_n^! \underset{C_n^!}{\otimes} U_1/U_0)) \cong H^i(\phi(B_n^! \underset{C_n^!}{\otimes} U_1))$ for $i \neq 0$.It follows $H^i(\phi(B_n^! \underset{C_n^!}{\otimes} U_1))$ is of Z-torsion for all i. By induction $H^i(\phi(B_n^! \underset{C_n^!}{\otimes} M))$ is of Z-torsion for all i.

We have proved $\phi(B_n^! \underset{C_n^!}{\otimes} M) \in \mathcal{T}$. □

Lemma 16. *Let M be a $B_n^!$-module and assume there is an integer $n \geq 0$ such that $\Omega^n M = N$ has the following properties:*

The module N is weakly Koszul, it has a filtration: $N = U_p \supset U_{p-1} \supset ...U_1 \supset U_0$ such that U_i/U_{i-1} is Koszul, and for all $k \geq 0$, $J^k U_i \cap U_{i-1} = J^k U_{i-1}$.

The Koszul modules $G_{B_n^!}(U_i/U_{i-1}) = V_i$ are of Z-.torsion.

Then $\phi(M)$ is in $\mathcal{T}$.

Proof. As above, $\phi(M)$ is in $\mathcal{T}$ if and only if $\phi(N)$ is in $\mathcal{T}$.

The exact sequence: $0 \to U_0 \to U_1 \to U_1/\ U_0 \to 0$ induces an exact sequence: $0 \to \phi(\ U_1/\ U_0) \to \phi(U_1) \to \phi(U_0) \to 0$ such that $H^0(\phi(U_1)) \cong H^0(\phi(U_0)) \cong G_{B_n^!}(U_0)$ is of Z-torsion and $H^i(\phi(\ U_1/\ U_0)) \cong H^i(\phi(U_1))$ is of Z-torsion for all $i \neq 0$. By induction, $H^i(\phi(N))$ is of Z-torsion for all i,hence $\phi(N)$ is in $\mathcal{T}$. □

Theorem 5. *Let $\mathcal{T}\ '$ be the subcategory of $\underline{gr}_{B_n^!}$ corresponding to* T *under the duality: $\overline{\phi} : \underline{gr}_{B_n^!} \to D^b(Qgr_{B_n^{op}})$. This is: $\overline{\phi}(\mathcal{T}\ ')$ =*T *.Then $\mathcal{T}\ '$ is the smallest triangulated subcategory of $\underline{gr}_{B_n^!}$ containing the induced modules and closed under the Nakayama automorphism.*

Proof. Let $\mathcal{B}$ be a triangulated subcategory of $\underline{gr}_{B_n^!}$ containing the induced modules and closed under the Nakayama automorphism. Let $M \in \mathcal{T}\,'$ and $\Omega^n M = N$ weakly Koszul with a filtration $N = U_p \supset U_{p-1} \supset ...U_1 \supset U_0$ such that U_i/U_{i-1} is Koszul, and for all $k \geq 0$, $J^k U_i \cap U_{i-1} = J^k U_{i-1}$.

Since $\mathcal{T}\,'$ is closed under the shift the module N is also in $\mathcal{T}\,'$. We prove by induction on the length of the filtration that for each i the modules U_i, $U_i/U_{i-1}$are in $\mathcal{T}\,'$.

We have an exact sequence of complexes: $0 \to \phi(\ N/\ U_0) \to \phi(N) \to \phi(U_0) \to 0$

By the long homology sequence there is an exact sequence:

$...H_{i+1}(\phi(U_0) \to H_i(\phi(\ N/\ U_0)) \to H_i(\phi(N)) \to H_i(\phi(U_0)) \to H_{i-1}(\phi(\ N/\ U_0))... \to H_0(\phi(\ N/\ U_0)) \to H_0(\phi(N)) \to H_0(\phi(U_0)) \to 0$

By $[MZ]$, $H_0(\phi(N)) = H_0(\phi(U_0))$, $H_0(\phi(\ N/\ U_0)) = 0$ and $H_i(\phi(U_0)) = 0$ for all $i \neq 0$, . Then $H_i(\phi(\ N/\ U_0)) = H_i(\phi(N))$ for all $i \neq 0$ and $H_0(\phi(U_0))$ is of Z-torsion and $H_i(\phi(\ N/\ U_0))$ is of Z -torsion for all i.

It follows by induction, U_i, $U_i/U_{i-1}$are in $\mathcal{T}\,'$ for all i.

The Koszul modules $G_{B_n^!}(U_i/U_{i-1}) = V_i$ are of Z-torsion and each $Z^j V_i$ is Koszul.

There exists an exact sequence:

$0 \to F_{B_n}(Z^{k_i} V_i)[-k_i] \to F_{B_n}(Z^{k_i-1} V_i/Z^{k_i} V_i)[-k_i+1]...$

$\to F_{B_n}(V_i/ZV_i) \to U_i/U_{i-1} \to 0$ where each $F_{B_n}(Z^j V_i/Z^{j+1} V_i) \cong B_n^! \underset{C_n^!}{\otimes} X_{ij}$ is an induced module of a Koszul $C_n^!$-module X_{ij}.

Then each $F_{B_n}(Z^j V_i/Z^{j+1} V_i) \cong B_n^! \underset{C_n^!}{\otimes} X_{ij}$ is in $\mathcal{B}$.

Moreover, the exact sequences: $0 \to \mathrm{B}_n^! \underset{C_n^!}{\otimes} \mathrm{X}_{ik_i} \to \mathrm{B}_n^! \underset{C_n^!}{\otimes} \mathrm{X}_{ik_i-1} \to \mathrm{K}_{k_i-2} \to 0$ gives rise to triangles: $B_n^! \underset{C_n^!}{\otimes} X_{ik_i} \to B_n^! \underset{C_n^!}{\otimes} X_{ik_i-1} \to K_{k_i-2} \to \Omega^{-1}(B_n^! \underset{C_n^!}{\otimes} X_{ik_i})$. Therefore $K_{k_i-2} \in \mathcal{B}$. It follows by induction, $U_i/U_{i-1} \in \mathcal{B}$.

The filtration $N = U_p \supset U_{p-1} \supset ...U_1 \supset U_0$ induces triangles: $U_0 \to U_1 \to U_1/U_0 \to \Omega^{-1}(U_0)$ with U_0,$U_1/U_0 \in \mathcal{B}$. It follows $U_1 \in \mathcal{B}$.

By induction, $N \in \mathcal{B}$.

We have proved $\mathcal{T}' \subset \mathcal{B}$. □

Theorem 6. *Let $\mathcal{T}\,'$ be the subcategory of $\underline{gr}_{B_n^!}$ corresponding to* T *under the duality: $\overline{\phi} : \underline{gr}_{B_n^!} \to D^b(Qgr_{B_n^{op}})$. This is: $\overline{\phi}(\mathcal{T}\,')$ =*T *. Then $\mathcal{T}\,'$has Auslander Reiten triangles and they are of type ZA_∞.*

Proof. Let M be an indecomposable non projective module in $\mathcal{T}\,'$. Then we have almost split sequences: $0 \to \sigma\Omega^2 M \to E \to M \to 0$ and $0 \to M \to F \to \sigma^{-1}\Omega^{-2} M \to 0$, since the category T is closed under the Nakayama automorphism, $\mathcal{T}\,'$ is also closed under the Nakayama automorphism and $\sigma\Omega^2 M$, $\sigma^{-1}\Omega^{-2} M$ are objects in $\mathcal{T}\,'$. From the exact sequences of complexes: $0 \to \phi(M) \to \phi(\ E) \to \phi(\sigma\Omega^2 M) \to 0$ and $0 \to \phi(\sigma^{-1}\Omega^{-2} M) \to \phi(F) \to \phi(M) \to 0$ and the long homology sequence we get that both $\phi(\ E)$ and $\phi(\ F)$ are in T. Therefore: E and F are in $\mathcal{T}\,'$. We have proved $\mathcal{T}\,'$has almost split sequences and they are almost split sequences in $gr_{_{B_n^!}}$. We proved in [MZ] that the Auslander Reiten components of $\underline{gr}_{B_n^!}$ are of type ZA_∞. It follows $\sigma\Omega^2 M \to E \to M \to \sigma\Omega^2 M[-1]$

and $M \to F \to \sigma^{-1}\Omega^{-2}M \to M[-1]$ are Auslander Reiten triangles and that the Auslander Reiten components are of type ZA_∞. $\square$

We will characterize now the full subcategory $\mathcal{F}'$ of $\underline{gr}_{B_n^!}$ such that $\overline{\phi}(\mathcal{F}') = \mathcal{F}$.

Theorem 7. *The subcategory $\mathcal{F}'$ of $\underline{gr}_{B_n^!}$ such that $\overline{\phi}(\mathcal{F}') = \mathcal{F}$ consists of the graded $B_n^!$-modules M such that the restriction of M to $C_n^!$ is injective.*

Proof. Let $M \in \mathcal{F}'$. There is an isomorphism: $Hom_{D^b(Qgr_{B_n^{op}})}(\mathcal{T},\ \pi\phi(M)) \cong \underline{Hom}_{grB_n^!}(M, \mathcal{T}') = 0$, which implies $\underline{Hom}_{B_n^!}(M, \mathcal{T}') = 0$

In particular for any induced module $B_n^! \underset{C_n^!}{\otimes} \Omega^2 L$ we have:

$\underline{Hom}_{B_n^!}(M, B_n^! \underset{C_n^!}{\otimes} \Omega^2 L) = 0 = \underline{Hom}_{B_n^!}(\Omega^{-2}M, B_n^! \underset{C_n^!}{\otimes} L)$.

By Auslander-Reiten formula:

$D(\underline{Hom}_{B_n^!}(\Omega^{-2}M, B_n^! \underset{C_n^!}{\otimes} L)) = Ext^1_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} L, M) = 0$ for all $L \in gr_{C_n^!}$.

Consider the exact sequences: $0 \to \Omega_{C_n^!}(L) \to F \to L \to 0$, with F the projective cover of L.It induces an exact sequence:

$0 \to B_n^! \underset{C_n^!}{\otimes} \Omega_{C_n^!}(L) \to B_n^! \underset{C_n^!}{\otimes} F \to B_n^! \underset{C_n^!}{\otimes} L \to 0$

By the long homology sequence, there is an exact sequence:

$0 \to Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} L, M) \to Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} F, M) \to Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} \Omega_{C_n^!}(L), M) \to Ext^1_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} L, M) \to 0$ which by the adjunction isomorphism are isomorphic to the exact sequences: $0 \to Hom_{C_n^!}(L, M) \to Hom_{C_n^!}(F, M) \to Hom_{C_n^!}(\Omega_{C_n^!}(L), M) \to Ext^1_{C_n^!}(L, M) \to 0$.

It follows, $Ext^1_{C_n^!}(L, M) \cong Ext^1_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} L, M)$ and by dimension shift

$Ext^k_{C_n^!}(L, M) \cong Ext^k_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} L, M) = 0$ for all $k \geq 1$.

We have proved the restriction of M to $C_n^!$ is injective.

Let's assume now the restriction of M to $C_n^!$ is injective:

Then for any integer n the restriction of $\Omega^n M$ to $C_n^!$ is injective.

Let $X \in \mathcal{T}'$. There exists an integer $n \geq 0$ such that $\Omega^n X = Y$, is weakly Koszul and it has a filtration: $Y = U_p \supset U_{p-1} \supset ...U_1 \supset U_0$ such that U_i/U_{i-1} is Koszul, and for all $k \geq 0$, $J^k U_i \cap U_{i-1} = J^k U_{i-1}$.The Koszul modules $G_{B_n^!}(U_i/U_{i-1}) = V_i$ are of Z-.torsion and each $Z^j V_i$ is Koszul.

Set $N = \Omega^n M$, the restriction of N to $C_n^!$ is injective.

There exist exact sequences:

$0 \to F_{B_n}(Z^{k_i}V_i)[-k_i] \to F_{B_n}(Z^{k_i-1}V_i/Z^{k_i}V_i)[-k_i+1]...$

$\to F_{B_n}(V_i/ZV_i) \to U_i/U_{i-1} \to 0$ where each $F_{B_n}(Z^j V_i/Z^{j+1}V_i) \cong B_n^! \underset{C_n^!}{\otimes} X_{ij}$ is an induced module of a Koszul $C_n^!$-module X_{ij}.

The exact sequences: $0 \to B_n^! \underset{C_n^!}{\otimes} X_{ik_i} \to B_n^! \underset{C_n^!}{\otimes} X_{ik_i-1} \to K_{k_i-2} \to 0$ induce exact sequences:

$0 \to Hom_{B_n^!}(K_{k_i-2}, N) \to Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} X_{ik_i-1}, N) \to Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} X_{ik_i}, N)$

$\to Ext^1_{B_n^!}(K_{k_i-2}, N) \to Ext^1_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} X_{ik_i-1}, N) \to Ext^1_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} X_{ik_i}, N)$

$\to Ext^2_{B_n^!}(K_{k_i-2}, N) \to Ext^2_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} X_{ik_i-1}, N) \to ..$

where $Ext^j_{B^!_n}(\ B^!_n \underset{C^!_n}{\otimes} X_{ik_i-l}, N) \cong Ext^j_{C^!_n}(X_{ik_i-1}, N) = 0$ for all $j \geq 1$.

It follows: $Ext^j_{B^!_n}(K_{k_i-2}, N) = 0$ for all $j \geq 2$.But $Ext^j_{B^!_n}(K_{k_i-2}, N)$ $\cong Ext^{j-1}_{B^!_n}(K_{k_i-2}, \Omega^{-1}N)$ and $Ext^j_{B^!_n}(K_{k_i-2}, \Omega^{-1}N) = 0$ for all $j \geq 1$.

The sequences: $0 \to K_{k_i-2} \to B^!_n \underset{C^!_n}{\otimes} X_{ik_i-2} \to K_{k_i-3} \to 0$ induce exact sequences:

$Ext^j_{B^!_n}(K_{k_i-3}, \Omega^{-1}N) \to Ext^j_{B^!_n}(\ B^!_n \underset{C^!_n}{\otimes} X_{ik_i-2}, \Omega^{-1}N) \to Ext^j_{B^!_n}(K_{k_i-2}, \Omega^{-1}N)$ $\to Ext^{j+1}_{B^!_n}(K_{k_i-3}, \Omega^{-1}N) \to Ext^{j+1}_{B^!_n}(\ B^!_n \underset{C^!_n}{\otimes} X_{ik_i-2}, \Omega^{-1}N) \to ..$

Therefore $Ext^{j+1}_{B^!_n}(K_{k_i-3}, \Omega^{-1}N) = 0$ for $j \geq 1$ which implies $Ext^j_{B^!_n}(K_{k_i-3}, \Omega^{-2}N) = 0$ for $j \geq 1$.

Continuing by induction there exist some $m \geq 0$ such that $Ext^j_{B^!_n}(U_i/U_{i-1}, \Omega^{-m}N) = 0$ for $j \geq 1$.

By induction on p we obtain $Ext^j_{B^!_n}(Y, \Omega^{-m}N) = 0$ for $j \geq 1$,in particular $Ext^1_{B^!_n}(Y, \Omega^{-m}N) = 0$.

By Auslander-Reiten formula, $Ext^1_{B^!_n}(Y, \Omega^{-m}N) \cong D(\underline{Hom}_{B^!_n}(\Omega^{-m}N, \Omega^2 Y)) \cong D(\underline{Hom}_{B^!_n}(N, \Omega^{2+m}Y)) \cong D(\underline{Hom}_{B^!_n}(\Omega^n M, \Omega^{2+m}Y))$.

It follows $\underline{Hom}_{B^!_n}(\Omega^n M, \Omega^{2+m+n}X) = 0$ which implies $\underline{Hom}_{B^!_n}(M, \Omega^{2+m}X) = 0$.

Observe m depends only on X. Taking $\Omega^{2+m}M$ instead of M we obtain $\underline{Hom}_{B^!_n}(\Omega^{2+m}M, \Omega^{2+m}X) = \underline{Hom}_{B^!_n}(M, X) = 0$.

Therefore $\underline{Hom}_{grB^!_n}(M, X) = 0$. It follows $M \in \mathcal{F}'$. □

Theorem 8. *The category $\mathcal{F}'$ is closed under the Nakayama automorphism, $\mathcal{F}'$ has Auslander Reiten sequences and they are of the form ZA_∞. Moreover, $\mathcal{F}'$ is a triangulated category with Auslander-Reiten triangles and they are of type ZA_∞.*

Proof. Let M be an indecomposable non projective object in $\mathcal{F}'$ and $0 \to \Omega M \to P \to M \to 0$ exact with P the projective cover of M.Since the restriction of P to $C^!_n$ is projective and restriction is an exact functor, it follows ΩM is in $\mathcal{F}'$. Similarly, $\Omega^{-1}M$ is in $\mathcal{F}'$.

If P is a projective $B^!_n$ -module and σ the Nakayama automorphism, then σP is also projective. Therefore: $\mathcal{F}'$ is closed under the Nakayama automorphism.

It is clear now that $\mathcal{F}'$ has Auslander Reiten sequences and they are of the form ZA_∞, by [MZ].

Let $f : M \to N$ be a homomorphism with M, N in $\mathcal{F}'$ and let $j : M \to P$ be the injective envelope of M. There is an exact sequence: $0 \to M \to P \oplus N \to L \to 0$ with M and $P \oplus N$ in $\mathcal{F}'$. Then L is also in $\mathcal{F}'$ and the triangle $M \to N \to L \to \Omega^{-1}M$ is a triangle in $\mathcal{F}'$. □

We have characterized the pair $(\mathcal{F}', \mathcal{T}')$ corresponding to $(\mathcal{T}, \mathcal{F})$ under the duality $\overline{\phi} : \underline{gr}_{B^!_n} \to D^b(Qgr_{B^{op}_n})$. Applying the usual duality $D : \underline{gr}_{B^!_n} \to \underline{gr}_{B^{!op}_n}$ we obtain a pair $(D(\mathcal{T}'), D(\mathcal{F}'))$ which corresponds to $(\mathcal{T}, \mathcal{F})$ under the equivalence: $\overline{\phi}D : \underline{gr}_{B^{!op}_n} \to D^b(Qgr_{B^{op}_n})$.

Observe the following:

From the bimodule isomorphism $B_n^! \sigma^{-1} \cong D(B_n^!)$, for any induced $B_n^!$-module $B_n^! \underset{C_n^!}{\otimes} X$, there are natural isomorphisms:

$Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} X, D(B_n^!)) \cong D(B_n^! \underset{C_n^!}{\otimes} X) \cong Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} X, B_n^! \sigma^{-1}) \cong$
$Hom_{C_n^!}(X, B_n^! \sigma^{-1}) \cong Hom_{C_n^!}(X, C_n^!) \underset{C_n^!}{\otimes} B_n^! \sigma^{-1}$.

For any finitely generated right $C_n^!$-module Y there exists a left $C_n^!$-module X such that $Hom_{C_n^!}(X, C_n^!) \cong Y$, hence $D(B_n^! \underset{C_n^!}{\otimes} X)\sigma \cong Y \underset{C_n^!}{\otimes} B_n^!$. Since $\mathcal{T}'$ is invariant under σ, $D(\mathcal{T}')$ is also invariant under σ and $D(\mathcal{T}')$ contains the induced modules.

Let B be a triangulated subcategory of $\underline{gr}_{B_n^{!op}}$ containing the induced modules. A triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} A[1]$ in $\underline{gr}_{B_n^!}$ comes from an exact sequence $0 \to A \overset{\begin{pmatrix} f \\ u \end{pmatrix}}{\to} B \oplus P \overset{(g,v)}{\to} C \to 0$ with P a projective module, hence $D(C) \overset{D(g)}{\to} D(B) \overset{D(f)}{\to} D(A) \to D(C)[1]$ is a triangle in $\underline{gr}_{B_n^{!op}}$. Therefore: $D(B)$ is a triangulated category containing the duals of the induced modules $D(Y \underset{C_n^!}{\otimes} B_n^!) \cong D(\ D(B_n^! \underset{C_n^!}{\otimes} X)\sigma) \cong D(Hom_{C_n^!}(X, C_n^!) \underset{C_n^!}{\otimes} B_n^!) \cong Hom_{B_n^!}(\ Hom_{C_n^!}(X, C_n^!) \underset{C_n^!}{\otimes} B_n^!, \sigma B_n^!) \cong \sigma B_n^! \underset{C_n^!}{\otimes} X$. Clearly $\sigma D(B)$ is a triangulated category containing the induced modules. Therefore: $\mathcal{T}' \subset \sigma D(B)$. Since $\mathcal{T}'$ is closed under Nakayama´s automorphism σ, $\mathcal{T}' \subset D(B)$. It follows $D(\mathcal{T}') \subset B$ and $D(\mathcal{T}')$ can be described as the smallest triangulated subcategory of $\underline{gr}_{B_n^{!op}}$ that contains the induced modules.

The usual duality D induces an isomorphism:
$D : Ext^i_{C_n^!}(M, N) \to Ext^i_{C_n^{op!}}(D(N), D(M))$.

It follows that the restriction of M to $C_n^!$ is injective if and only if the restriction of $D(M)$ to $C_n^{op!}$ is projective (injective). It follows $D(\mathcal{F}')$ is the category of $B_n^{op!}$-modules whose restriction to $C_n^{op!}$ is injective.

T=$D(\mathcal{T}')$ is a "epasse" subcategory of $\underline{gr}_{B_n^{!op}}$.The functor $\overline{\phi}D$ induces an equivalence of categories: $\underline{gr}_{B_n^{!op}}/T \cong D^b(Qgr_{B_n^{op}})/\mathcal{T}$ and we proved $D^b(Qgr_{B_n^{op}})/\mathcal{T} \cong D^b(gr_{(B_n)_Z})$.

The equivalence $gr_{(B_n)_Z} \cong \mathrm{mod}_{A_n}$ induces an equivalence: $D^b(gr_{(B_n)_Z}) \cong D^b(\mathrm{mod}_{A_n})$.

We have proved:

Theorem 9. *There is an equivalence of triangulated categories:*
$\underline{gr}_{B_n^!}/\mathrm{T} \cong D^b(\mathrm{mod}_{A_n})$.

The category F= $D(\mathcal{F}')$ is the category of all T -local objects it is triangulated. By [Mi], there is a full embedding: $F \to \underline{gr}_{B_n^{!op}}/T \cong D^b(\mathrm{mod}_{A_n})$.

Proposition 13. *The category $ind_{C_n^!}$ of all induced $B_n^!$-modules is contravariantly finite in $gr_{B_n^!}$.*

Proof. Let M be a $B_n^!$-module and $\mu : B_n^! \underset{C_n^!}{\otimes} M \to M$ the map given by multiplication. Let $\alpha : Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M, M) \to Hom_{C_n^!}(M, M)$ the morphism giving the adjunction. It is easy to see that $\alpha(\mu) = 1_M$.

Let $\varphi : B_n^! \underset{C_n^!}{\otimes} N \to M$ be any map and $\alpha(\varphi) = f : N \to M$ the map given by adjunction. There is a commutative square

$$\begin{array}{ccc} Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} M, M) & \overset{\alpha_M}{\to} & Hom_{C_n^!}(M, M) \\ \downarrow (1 \otimes f, M) & & \downarrow (f, M) \\ Hom_{B_n^!}(B_n^! \underset{C_n^!}{\otimes} N, M) & \overset{\alpha_N}{\to} & Hom_{C_n^!}(N, M) \end{array}$$

from the commutativity of the diagram $f = \alpha_N(\varphi) = \alpha_N(\mu 1 \otimes f)$ implies $\varphi = \mu 1 \otimes f$.

We have proved the triangle:
$$\begin{array}{ccc} & & B_n^! \underset{C_n^!}{\otimes} N \\ & 1 \otimes f \swarrow & \downarrow \phi \\ B_n^! \underset{C_n^!}{\otimes} M & \overset{\mu}{\to} & M \end{array}$$
commutes. □

Corollary 9. *add($ind_{C_n^!}$) is contravariantly finite.*

Corollary 10. *$ind_{C_n^!}$ is functorialy finite.*

Proof. It is clear from the duality $D(\mathrm{ind}_{C_n^!}) \cong \mathrm{ind}_{C_n^{op!}}$ □

Observe $(\mathrm{ind}_{C_n^!})^{\perp} \cong \mathcal{F}'$ however $\mathrm{ind}_{C_n^!}$ is not necessary closed under extensions and we can not conclude $\mathcal{F}'$ contravariantly finite.

For the notions of contravariantly finite, covariantly finite and functorialy finite, we refer to [AS].

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